Random walks: Step by Step

Armin Bunde
Justus-Liebig-Universität Giessen

Outline

(I) Random walks: Mean square displacement, density distribution

(II) The structure of a random walk

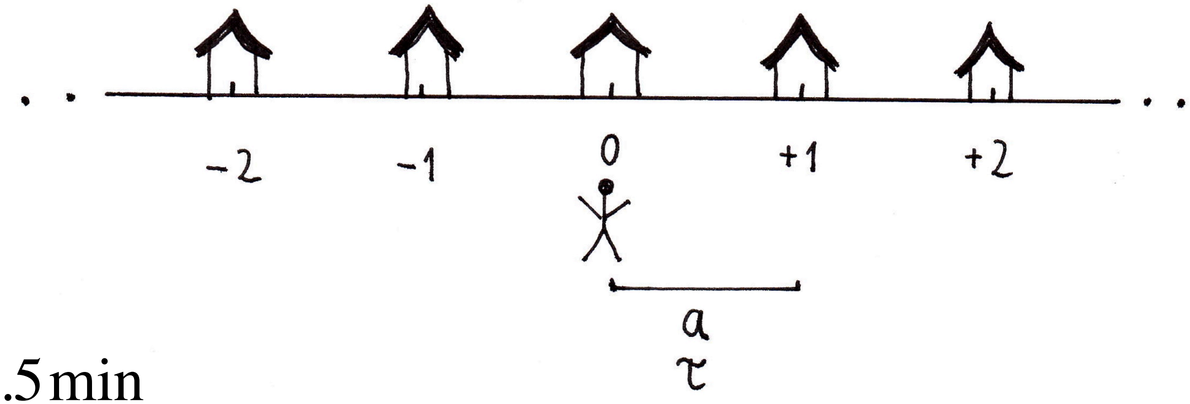
(III) Structures generated by random walks

(IV) Random walks in disordered media

(V) Many random walkers: Spreading phenomena

(I) Random walks: Mean square displacement, density distribution

Druncan Sailor Problem:



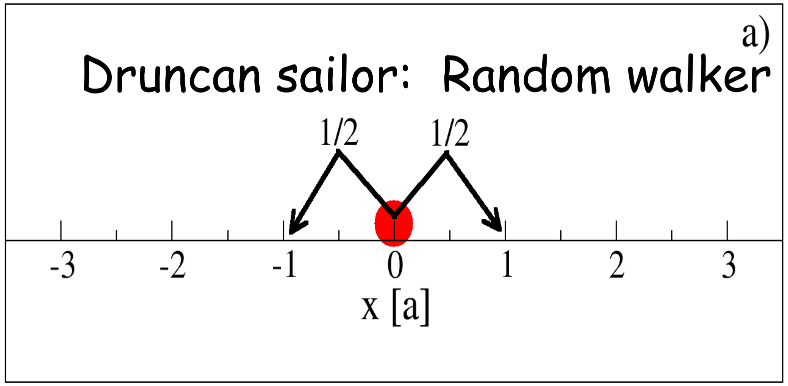
$a = 30m, \tau = 0.5\,\mathrm{min}$

(i) Boarding house is 300m away

(ii) Boarding house is 600m away

When will the druncan sailor arrive there?

Problem is equivalent to Gamblers Ruin (Jakob Bernoulli, 300y ago)



Position of the random walker after n steps:

$$\vec{x}(t = n\tau) = a\sum_{i=1}^{n}\vec{e_i}$$

$\downarrow$

$\vec{r}$ in d dimensional lattices

Mean-square displacement (average over many random walkers):

$$< x^2(t = n\tau) >= a^2 < (\sum_{i=1}^{n}\vec{e_i})(\sum_{j=1}^{n}\vec{e_j}) >$$

$$= a^2(< (\sum_{i=1}^{n}(\vec{e_i})^2 > + < (\sum_{i\neq j}^{n}\vec{e_i}\vec{e_j}) >)$$

$$= a^2 n = \frac{a^2}{\tau}t \equiv (2d)Dt$$

Diffusion law

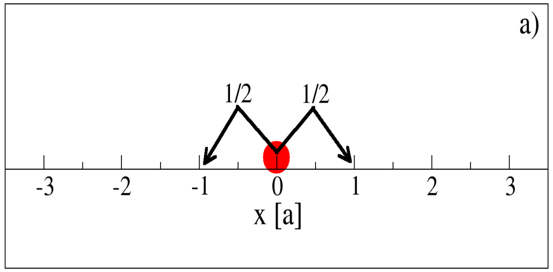
$$X = \sqrt{< x^2 >} \textit{ typical length travelled}$$

$$X^2 = \frac{a^2}{\tau}t \Rightarrow t = \frac{\tau}{a^2} X^2$$

$$X = 10a, \tau = 0.5\,\text{min} \rightarrow t = 50\,\text{min}$$

$$X = 20a, \tau = 0.5\,\text{min} \rightarrow t = 200\,\text{min}$$

Probability distribution



$$P_n(m) = \frac{1}{2} P_{n-1}(m-1) + \frac{1}{2} P_{n-1}(m+1), \qquad t = n\tau, \quad x = ma, \ P_n(m) = a\,P(x,t)$$

$$P(x,t) = \frac{1}{2} P(x-a, t-\tau) + \frac{1}{2} P(x+a, t-\tau)$$

$$P(x,t) - P(x, t-\tau) = \frac{1}{2} P(x-a, t-\tau) + \frac{1}{2} P(x+a, t-\tau) - P(x, t-\tau)$$

$$\frac{P(x,t) - P(x, t-\tau)}{\tau} = \frac{a^2}{2\tau} \frac{P(x-a, t-\tau) + P(x+a, t-\tau) - 2P(x, t-\tau)}{a^2}$$

$$\frac{\partial P}{\partial t} = \left(\frac{a^2}{2\tau}\right) \frac{\partial^2 P}{\partial x^2}$$

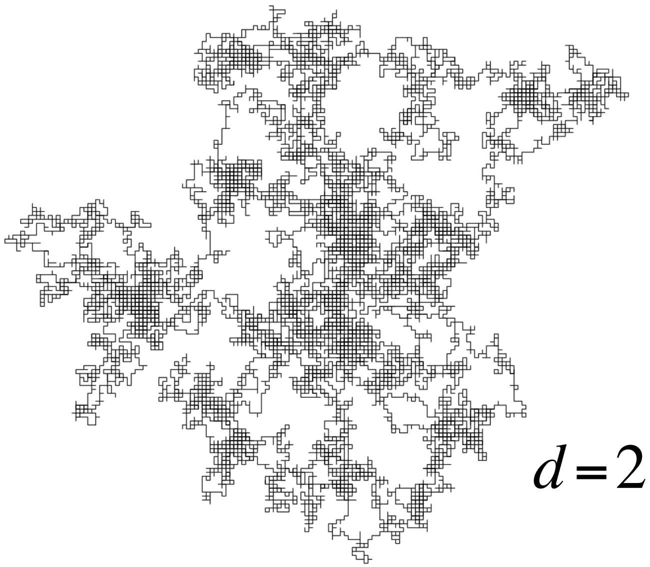
Diffusion equation

$$2D \hookleftarrow$$

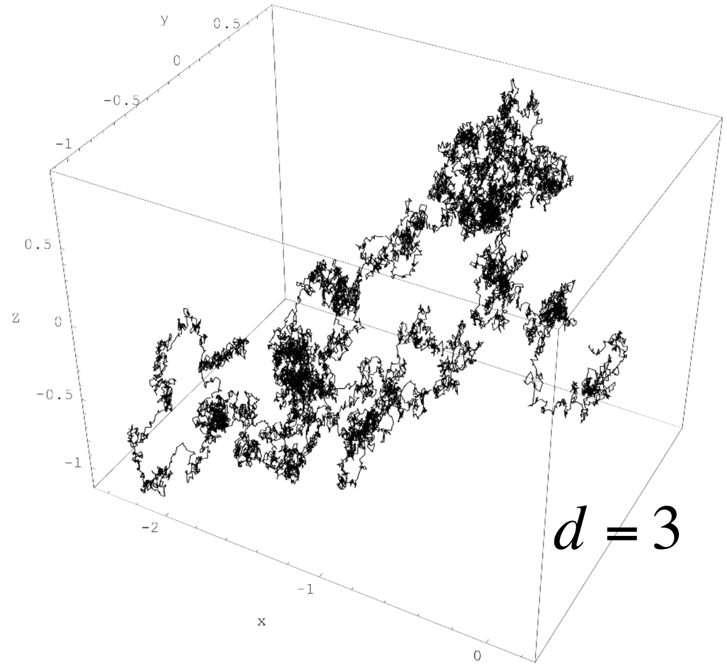
$$solution: P(x,t) = \frac{1}{(2\pi D t)^{1/2}} \exp(-x^2/4Dt) \equiv \frac{1}{\pi^{1/2} X(t)} \exp(-x^2/2X^2(t))$$

Gaussian with width X(t)

(II) Structure of random walks



$d=2$



$d=3$

What is the dimension of these structures?

$$M \propto t \propto R^2 \equiv R^{d_f} \Rightarrow d_f = 2$$ for all space dimensions d

“Hull” of the random walk in d=2: $d_f = 4/3$

(III) Structures generated by random walks

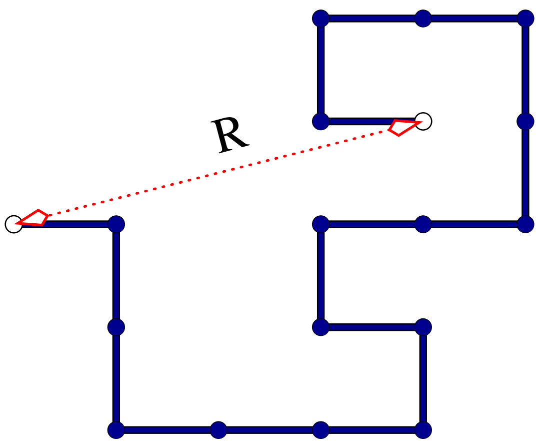
1. Self-Avoiding Walks: Model for linear polymers in dilute solution

A random walker that cannot go back

to a site he visited before.

Topological linear structure, similar

to the hull of the random walk.



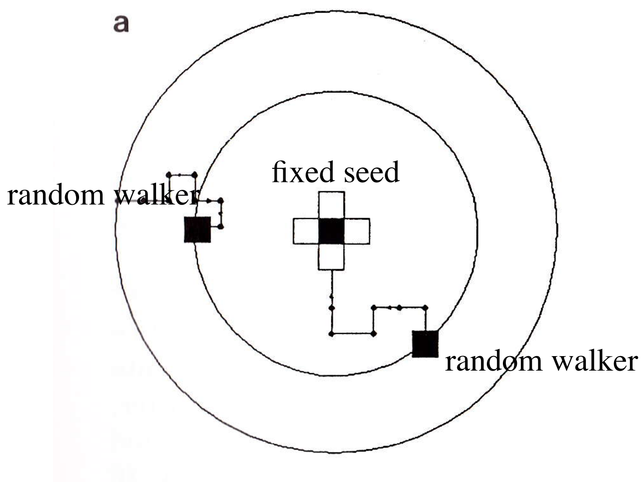
Fractal structure: $d_f = 4/3 \ (d = 2)$

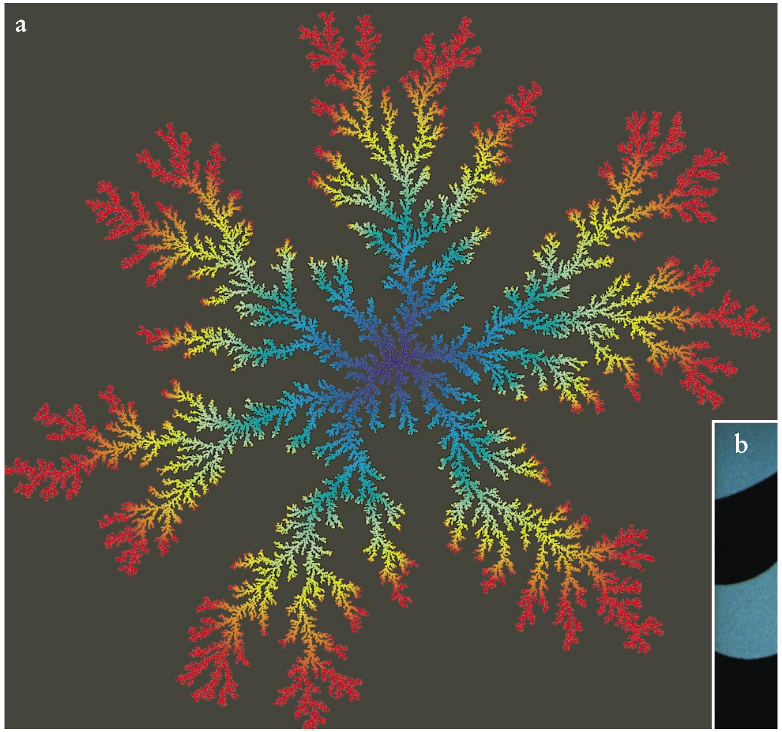
$d_f = 5/3 \ (d = 3)$

Flory, 1953

2. Diffusion limited aggregation

Model for aggregates, electrodeposition, dielectric breakdown...

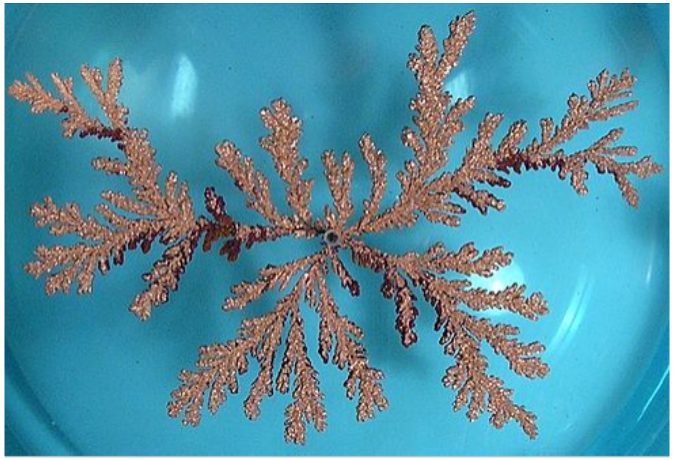




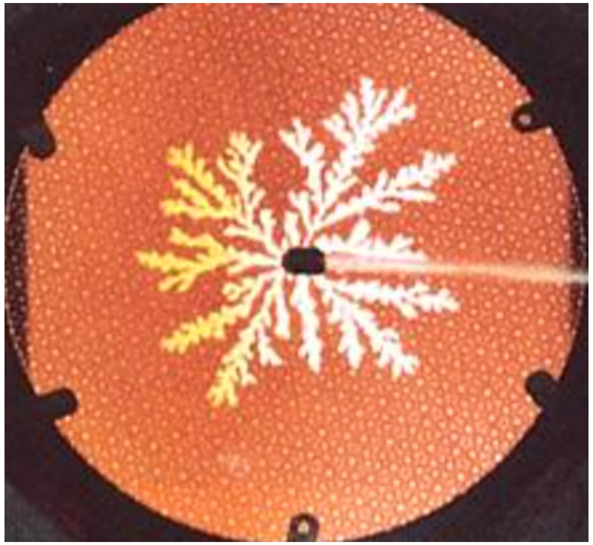
Fractal structure: $d_f \cong 1.7 \ (d = 2)$

$d_f \cong 2.5 \ (d = 3)$

Witten + Sander, 1981



Electrodeposition



Viscous flow

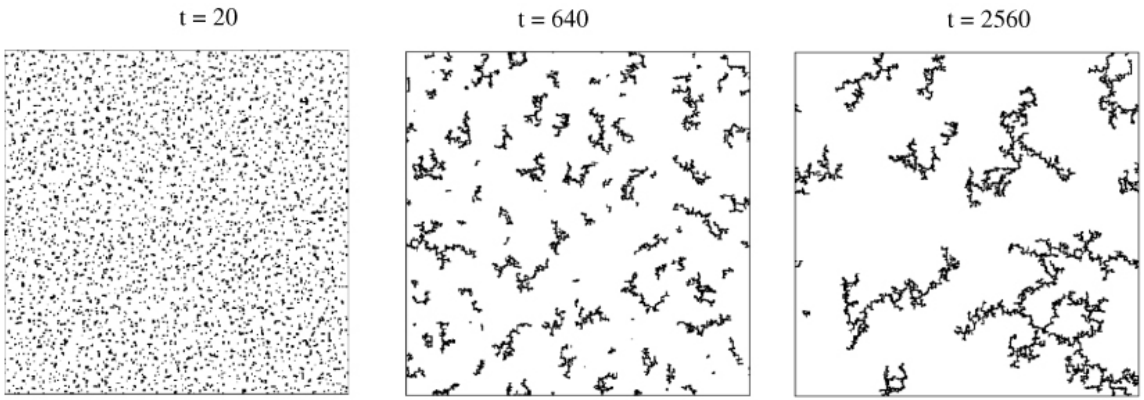
Dielectric Breakdown

Dried polymers

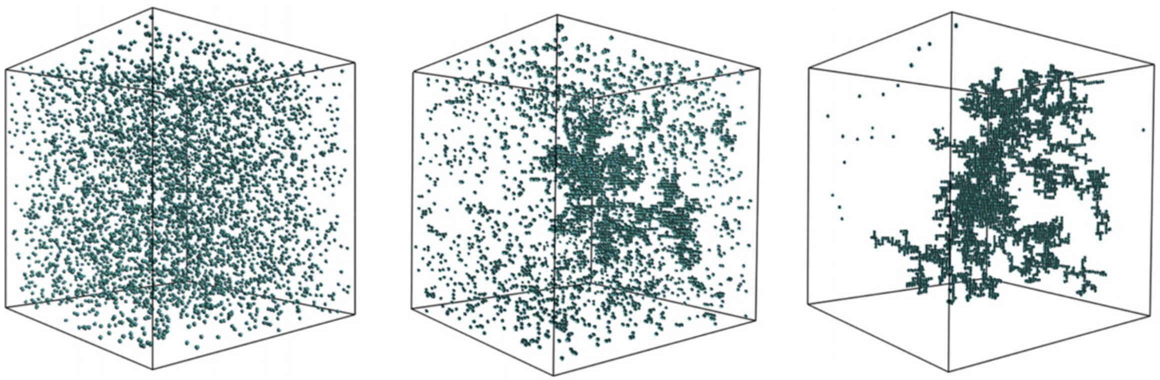
Dawn in the Himalayas

3. Cluster-cluster aggregation

Model for colloid aggregates (gold, silica, aerogels, smoke...)



Random walkers, stick together when on nearest neighbor sites



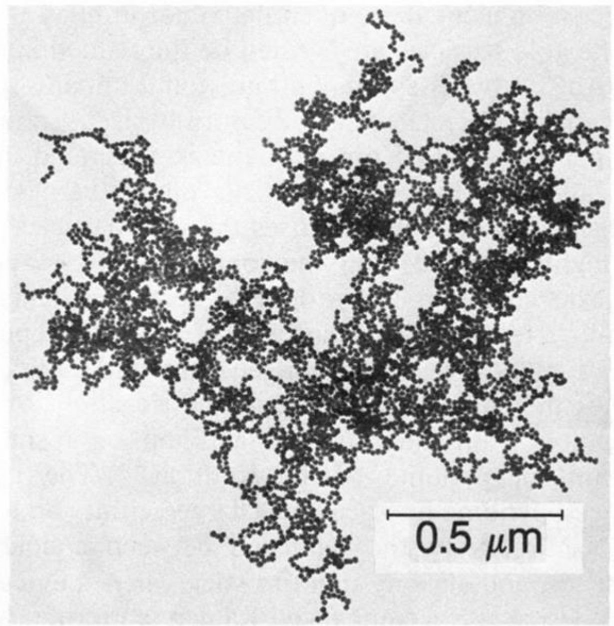
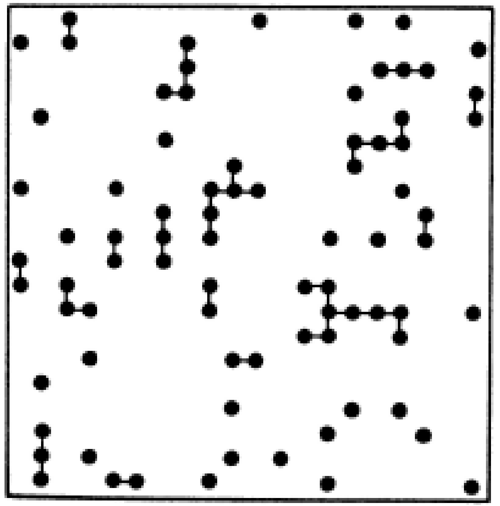


FIG. 1. TEM image of typical gold colloid aggregate. This cluster contains 4739 gold particles.

Fractal structure: $d_f \cong 1.8 \ (d = 3)$

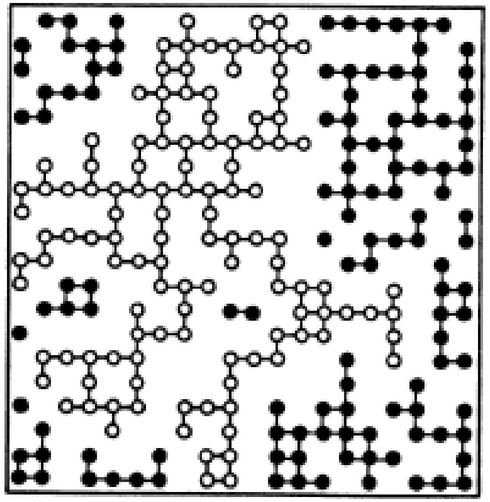
Weitz et al, 1987

(IV) Random walks in disordered media



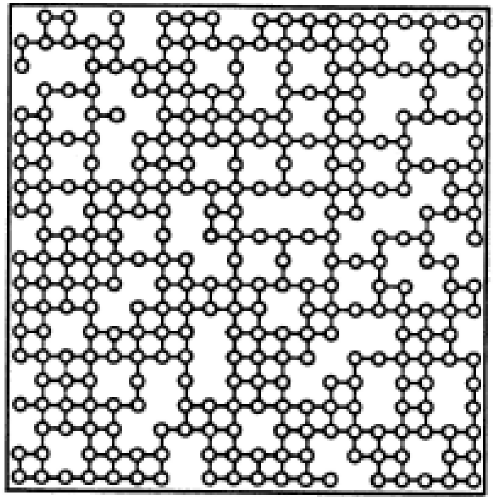
$p = 0.2$

$p < p_c$: finite clusters of occupied sites



$p = 0.59$

p_c: critical concentration: spanning ("infinite") cluster emerges



$p = 0.8$

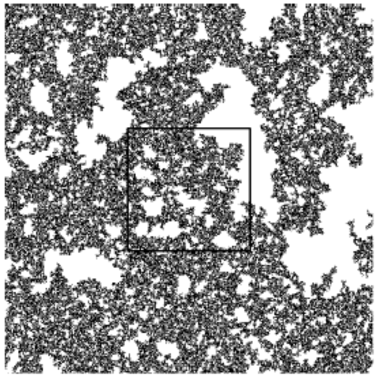
$p > p_c$: infinite cluster + finite clusters

mean length ξ of finite clusters: $\xi \sim (p - p_c)^{-\nu}$

size P_∞ of the infinite cluster: $P_\infty \sim (p - p_c)^{\beta}$

1. Fractal structures:

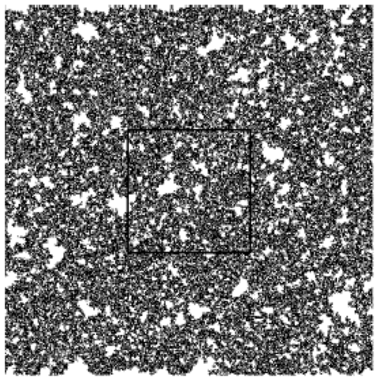
At p_c:



$$M_\infty \sim r^{d_f}$$

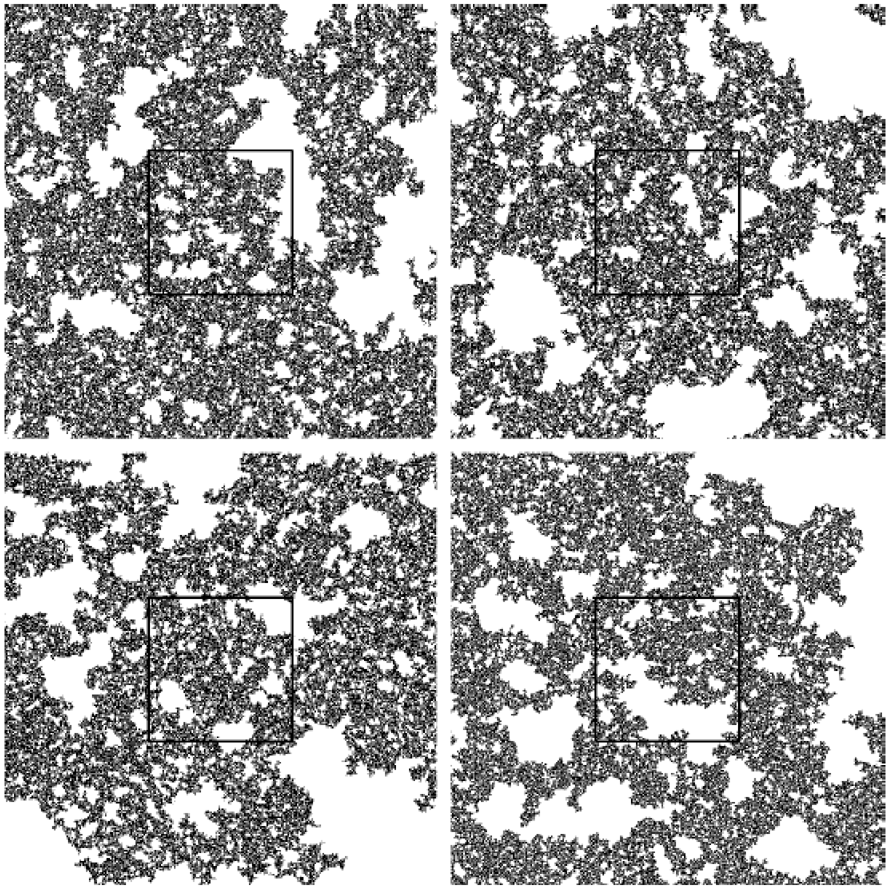
$$d_f \cong 1.96 \ (d = 2), \ \ d_f \cong 2.5 \ (d = 3)$$

Above p_c:

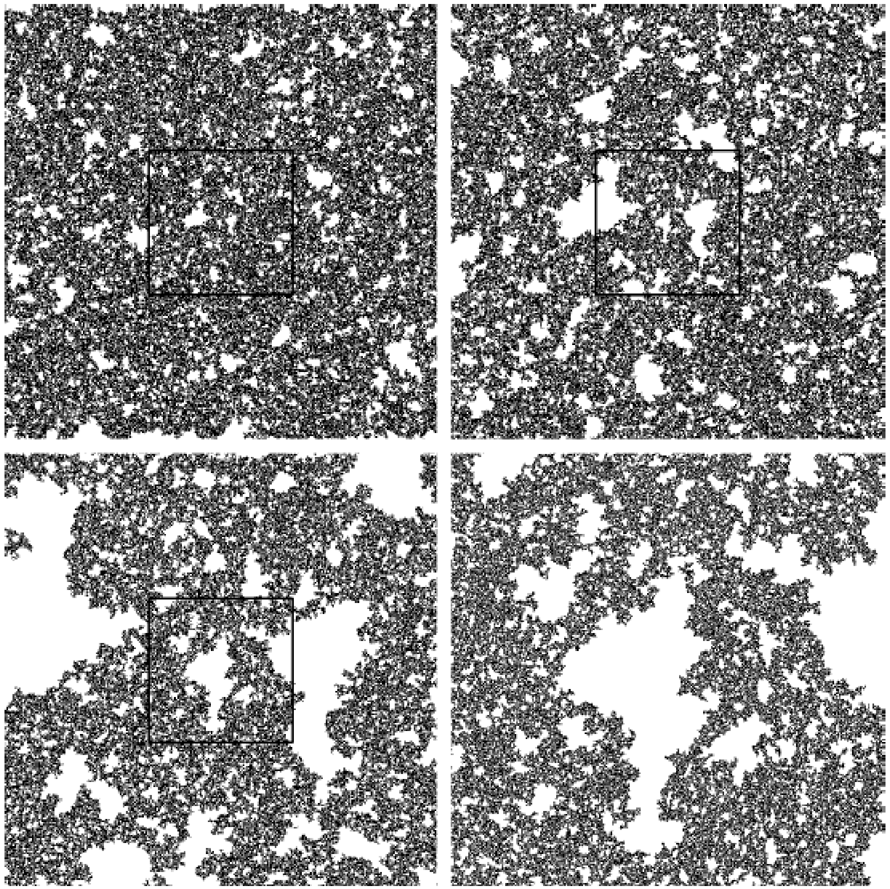


$$M_\infty \sim \begin{cases} r^{d_f}, \ r << \xi \\ r^{d}, \ r >> \xi \end{cases}$$

(a) Self-similarity at p_c:

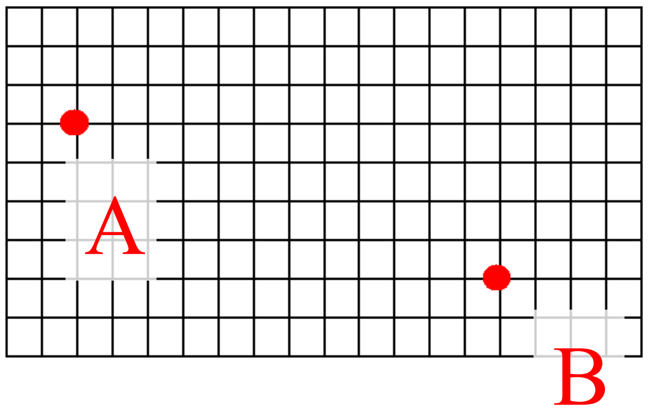


(b) Self-similarity above p_c:



2. Anomalous diffusion:

Normal lattice

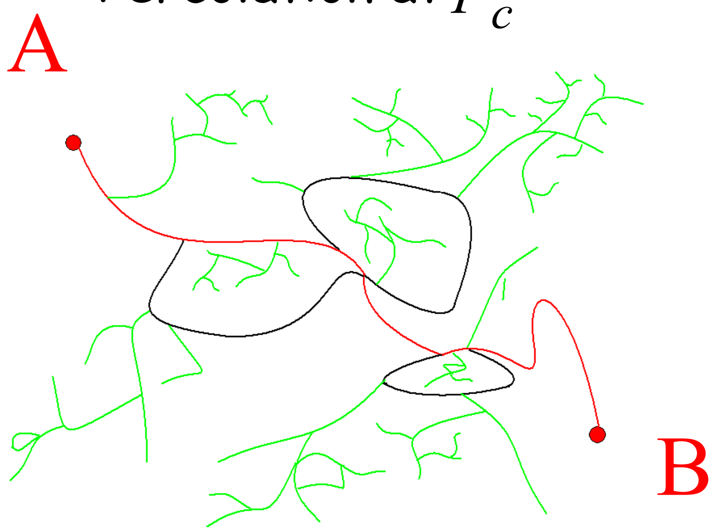


$$t_{AB} \sim R_{AB}^{\ 2}$$

$$\Downarrow$$

$$\left\langle r^2(t) \right\rangle \sim t$$

Percolation at p_c



$$t_{AB} \sim R_{AB}^{\ d_w}$$

$$\Downarrow$$

$$\left\langle r^2(t) \right\rangle \sim t^{2/d_w}$$

$d_w \cong 2.9 \ (d = 2), \ \ d_w \cong 3.7 \ (d = 3)$

Diffusion above p_c

$$\langle r^2(t)\rangle \sim \begin{cases} t^{2/d_w}, & t \ll t_\xi \sim \xi^{d_w} \\ t, & t \gg t_\xi \sim \xi^{d_w} \end{cases}$$

$$D = \left\langle r^2(t) \right\rangle \Big/ 2dt$$

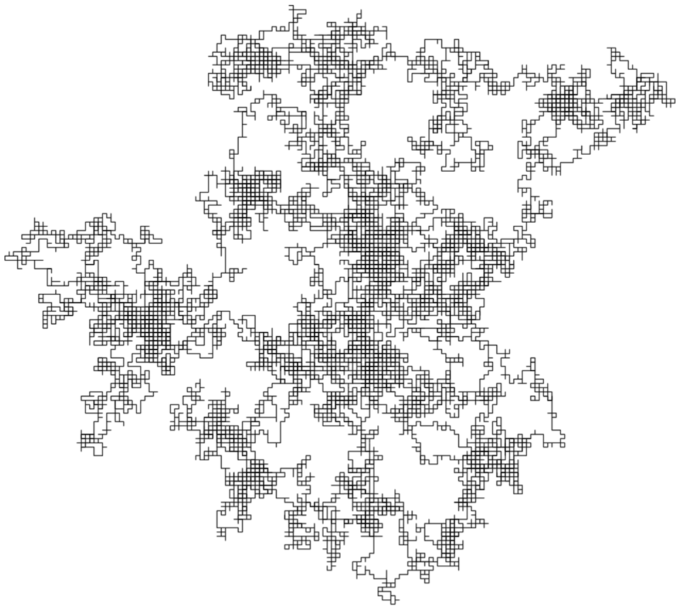
$$D \sim \begin{cases} (p-p_c)^{\mu}, & p > p_c \\ 0, & p \le p_c \end{cases}$$

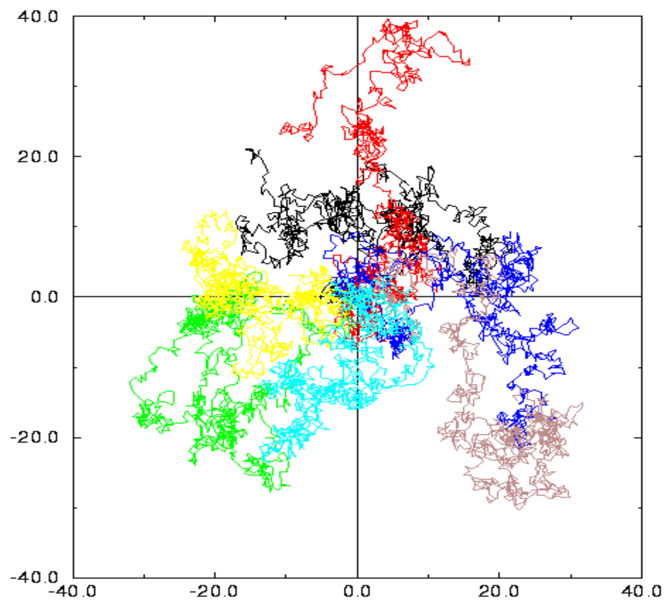
Relation between μ and d_w: $\mu = (d_w - 2)\nu$

Proof: $D \sim \langle r^2(t_\xi)\rangle / t_\xi \sim t_\xi^{2/d_w - 1} \sim \xi^{d_w(2/d_w - 1)} \sim \xi^{2-d_w} \sim (p-p_c)^{\nu(d_w-2)}$

(V) N random walkers: Spreading phenomena

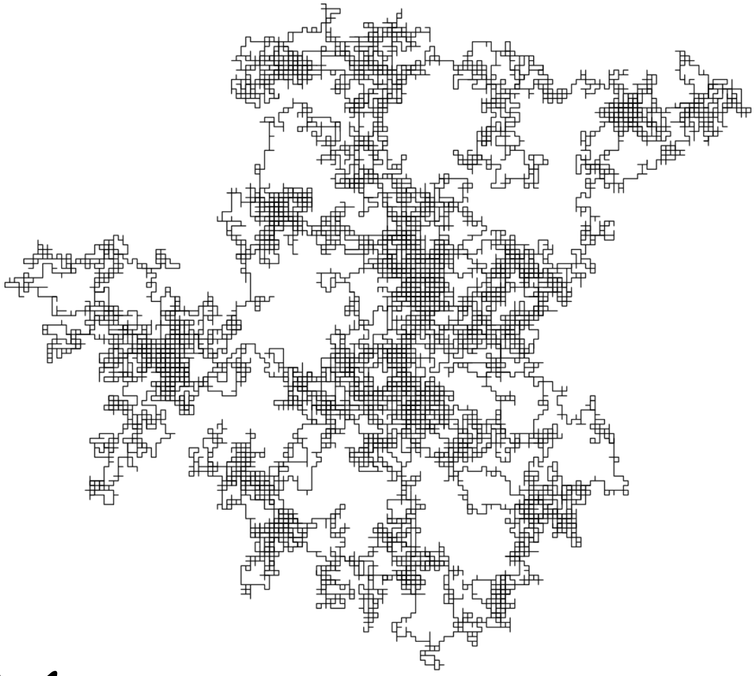
N=1





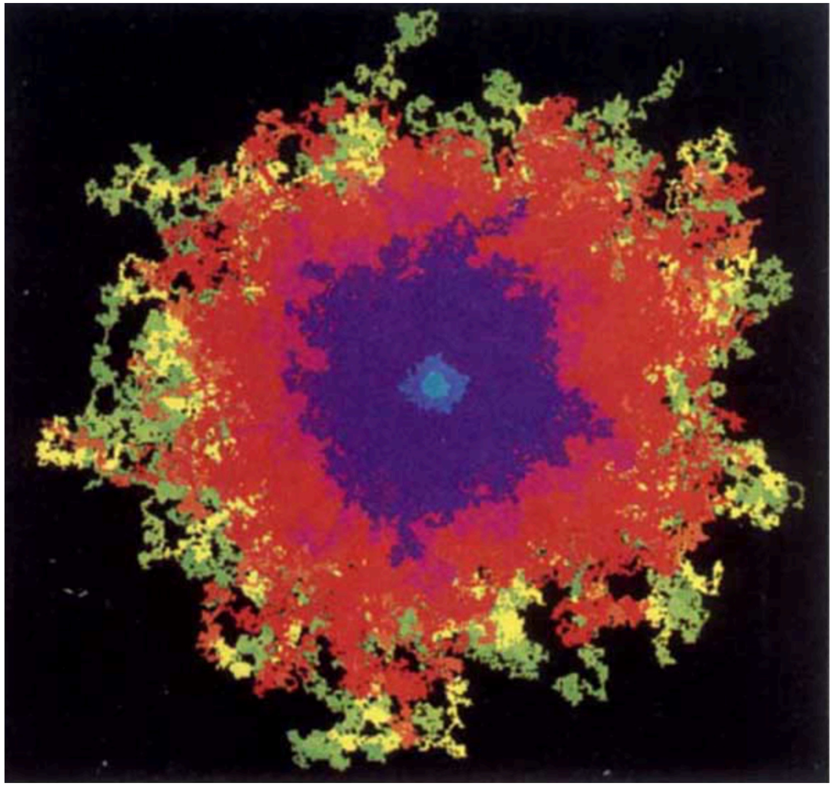
N=7

Number of distinct sites S_N visited ?



N=1

N=1000



Number of distinct sites visited:

$$S_1 \propto \begin{cases} t^{1/2} & (d=1) \\ t/\ln t & (d=2) \\ t & (d=3) \end{cases}$$

$$N >> 1: \ S_N \propto t^{d/2} \left(\ln(NS_1(t)/t^{d/2} \right)^{d/2}$$
$$= t \quad \ln(N/\ln t) \quad in \ d=2$$

Larralde et al, 1992