Rotation

two parts: 1) motion in two dimensions

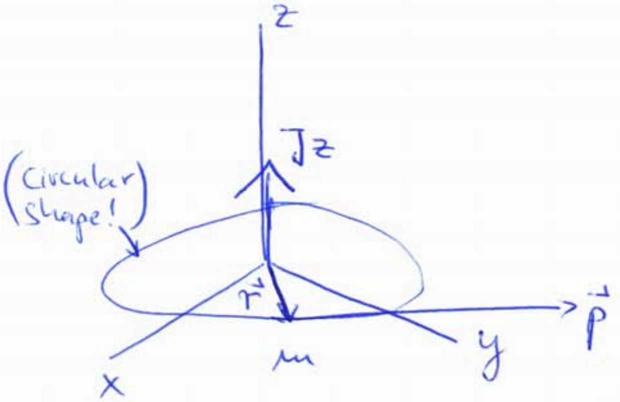
2) rotation in 3d

2d rotation, particle on a ring

- particle of mass m move in a circular path of radius r in the xy-plane.
- total energy = kinetic energy, because $V = 0$ everywhere

$\Rightarrow$ one may write: $E = p^2/2m$

classical picture



J_z = angular moment (around z-axis)

$J_z = \pm pr$

$\Rightarrow E = \dfrac{J_z^2}{2mr^2}$ with $mr^2 = I$

I = moment of inertia

- with the deBroglie wavelength ($\rightarrow$ transition to Q.M.)

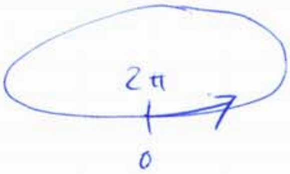
$p = h/\lambda$

$J_z = \pm \dfrac{hr}{\lambda}$ opposite signs correspond to opposite directions in travel

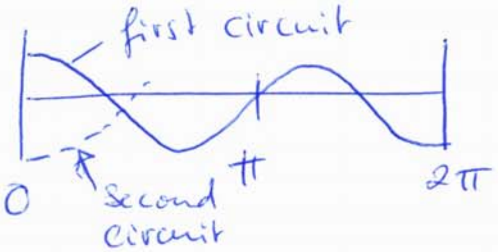
$\Rightarrow$ the shorter the wavelength, the larger the angular momentum

2d rotation: acceptable wavefunctions are sought:

λ (De Broglie) can take an arbitrary value



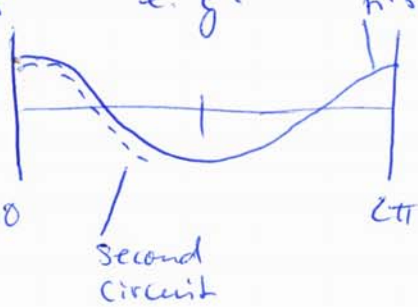
behavior of the wavefunction is dependent on the azimuthal angle ϕ



- ϕ increases beyond 2π, then the wavefunction would continue to change

  → different value at each point than in the first circuit (which is not acceptable)

Acceptable wavefunction: reproduces itself on different circuits e.g.



⟹ quantization, not all values of J_z are allowed

why? <u>Cyclic</u> <u>boundary condition</u> of selfreproduction of the rotator's wavefunction on successive circuits.

allowed wavelengths $\lambda = \dfrac{2\pi r}{m_l}$, q. numbers $m_l = 0, \pm 1, \pm 2 \ldots$

$\lambda = \infty$

angular momentum:

$$J_z = \pm \frac{hr}{\lambda} = \frac{m_l hr}{2\pi r} = m_l \hbar$$ ($\pm$: clockwise / counterclockwise direction)

$$E = \frac{J_z^2}{2I} = \frac{m_l \hbar^2}{2I}$$

wave functions:

$$\psi_{m_l}(\phi) = \frac{e^{-im_l\phi}}{(2\pi)^{1/2}}$$

with $m_l = 0 \Rightarrow \psi_0(\phi) = 1/(2\pi)^{1/2}$

Conclusions

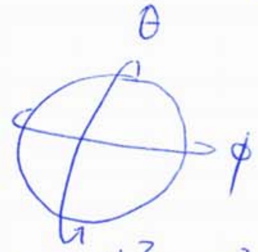
- quantized Energy, square of m_l: energy independent of the sense of rotation
- given $|m_l|$ states: doubly degenerate, $m_l = 0$ non-degenerate
- $J_z = m_l \hbar$ increasing $m_l \rightarrow$ increased number of nodes in the wave function $\rightarrow$ wavelength decreases stepwise, as $|m_l|$ increases $\Rightarrow$ momentum of the particle increases.
- locate the particle: form probability density

$$\psi^*_{m_l} \cdot \psi = \left(\frac{e^{im_l\phi}}{(2\pi)^{1/2}}\right)^* \cdot \left(\frac{e^{im_l\phi}}{(2\pi)^{1/2}}\right) = \frac{1}{2\pi}$$

$\Rightarrow$ probability density independent of ϕ, position of the particle = completely indefinite! (if angular momentum known)

Rotation in 3 dimensions

- particle on a sphere



$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

- 3d Schrödinger eq.: $H = \frac{\hbar^2}{2m}\nabla^2 + V$
- solving with usage of Legendre functions: Spherical harmonics
- wave function $\psi(\theta, \phi)$, colatitude θ, azimuth ϕ
- Schrödinger eq. can be simplified by separation of variables

$$\psi(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

Θ is function of θ only

Φ is function of ϕ only

$\rightarrow$ range of restricted m_l-values (by l)

$\underline{\underline{l}} = 0, 1, 2.$; at given l: $2l+1$ permitted values of the <u>magnetic</u> quantum number m_l

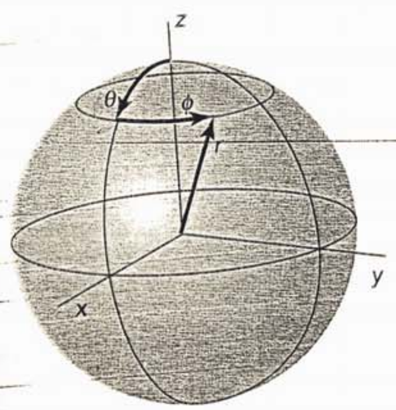
$m_l = 0, \pm 1, \pm 2 \ldots$

orbital / angular momentum quantum number

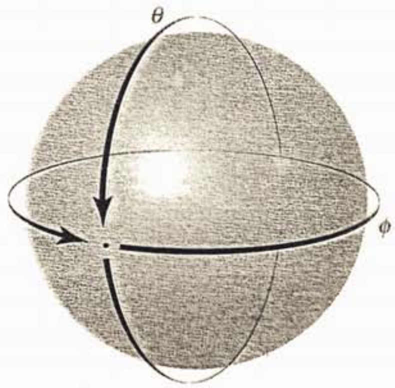
sperency

- angular momentum: $\sqrt{l(l+1)}\,\hbar$, $l = 0, 1, 2$
- z-component of the angular momentum: $m_l \hbar$

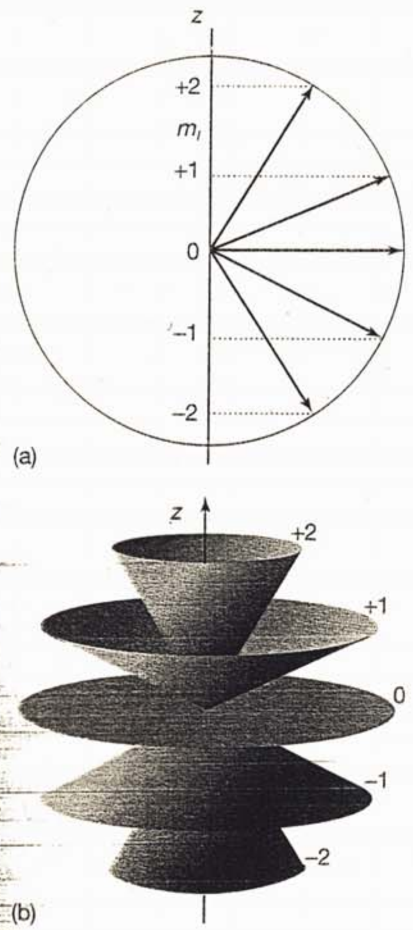
$\rightarrow$ <u>orientation</u> of rotating body: quantized $m_l = l, l-1 \ldots -l$



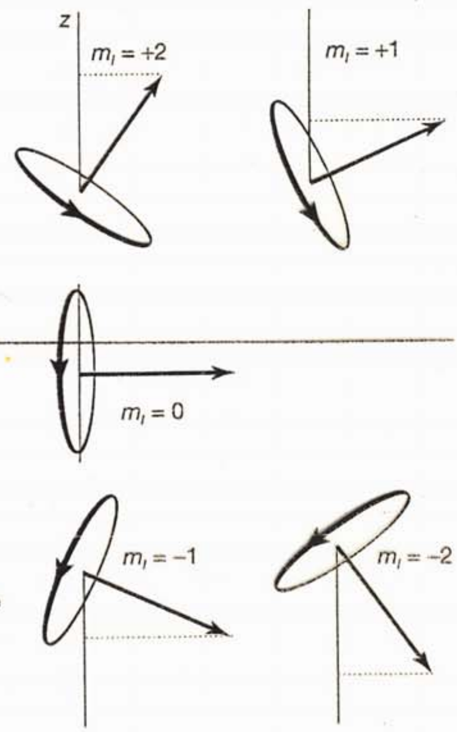
12.31 Spherical polar coordinates. For a particle confined to the surface of a sphere, only the colatitude, θ, and the azimuth, ϕ, can change.



12.30 The wavefunction of a particle on the surface of a sphere must satisfy two cyclic boundary conditions; this requirement leads to two quantum numbers for its state of angular momentum.



12.36 (a) A summary of Fig. 12.34. However, because the azimuthal angle of the vector around the z-axis is indeterminate, a better representation is as in (b), where each vector lies at an unspecified azimuthal angle on its cone.



12.34 The permitted orientations of angular momentum when $l = 2$. We shall see soon that this representation is too specific because the azimuthal orientation of the vector (its angle around z) is indeterminate.