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Quantum Field Theory of Many-Particle Systems - Problem Set 2

Winter Term 2011/12

Due Date: Wednesday, October 26, 4 p.m., tutorial session

Internet: You can download this problem set at http://www.uni-leipzig.de/∼rosenow.

1. Quantization of the Radiation Field

5+5+5 Punkte

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2} \int_\Omega d^3x \left[\epsilon_0 \left(\partial_t \mathbf{A}\right)^2 + \frac{1}{\mu_0} \mathbf{A} \nabla^2 \mathbf{A} \right] \quad .$$

Here, ϵ_0 denotes the vacuum dielectric constant, μ_0 the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

a) Derive the equation of motion for $\mathbf{A}$ by extrimizing the Lagrangian.

b) Find eigenfunctions $\mathbf{A_k}$ and eigenvalues $\omega_\mathbf{k}^2$ of the equation

$$-\nabla^2 \mathbf{A} = \frac{\omega_\mathbf{k}^2}{c^2} \mathbf{A} \quad .$$

It may be useful to introduce for each $\mathbf{k}$ a set of orthonormal vectors $\{\xi_{\mathbf{k},1}, \xi_{\mathbf{k},2}\}$ which are both perpendicular to $\mathbf{k}$. The time-dependent solution has then a series expansion

$$\mathbf{A}(\mathbf{x}, t) = \sum_\mathbf{k} \alpha_\mathbf{k}(t) \mathbf{A_k}(\mathbf{x}) \quad .$$

Insert this series expansion into the Lagrangian, and find the momenta

$$\pi_\mathbf{k} = \frac{\partial L}{\partial \dot{\alpha_\mathbf{k}}} \quad .$$

canonically conjugate to the coordinates $\alpha_\mathbf{k}$. Use the Legendre transform $H = \sum_\mathbf{k} \pi_\mathbf{k} \dot{\alpha}_\mathbf{k} - L(\pi_\mathbf{k}, \alpha_\mathbf{k})$ to obtain the Hamiltonian.

c) The classical Hamiltonian $H(\{\pi_\mathbf{k}, \alpha_\mathbf{k}\})$ can be quantized by imposing canonical commutation relations

$$[\alpha_\mathbf{k}, \alpha_\mathbf{q}] = 0, \quad [\pi_\mathbf{k}, \pi_\mathbf{q}] = 0 \,, \quad [\alpha_\mathbf{k}, \pi_\mathbf{q}] = i\hbar \, \delta_{\mathbf{k},\mathbf{q}} \quad .$$

on coordinates $\alpha_\mathbf{k}$ and canonically conjugate momenta $\pi_\mathbf{k}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a_\mathbf{k} = \sqrt{\frac{\epsilon_0 \omega_\mathbf{k}}{2\hbar}} \left(\alpha_\mathbf{k} + \frac{i}{\epsilon_0 \omega_\mathbf{k}} \pi_{-\mathbf{k}} \right) \quad , \quad a_\mathbf{k}^\dagger = \sqrt{\frac{\epsilon_0 \omega_\mathbf{k}}{2\hbar}} \left(\alpha_{-\mathbf{k}} - \frac{i}{\epsilon_0 \omega_\mathbf{k}} \pi_\mathbf{k} \right)$$

Show that the $a_\mathbf{k}$, $a_\mathbf{k}^\dagger$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of the $a_\mathbf{k}$, $a_\mathbf{k}^\dagger$.