Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow

Quantum Field Theory of Many-Particle Systems - Problem Set 7

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Internet: You can download this problem set at http://www.uni-leipzig.de/∼rosenow.

10. Charge neutrality and zero momentum component *8 Punkte*

In the Jellium model, the negative charge of the electrons is compensated by a positively charged background of ions, which is assumed to be spatially homogeneous. Show that this positive background is responsible for the absence of the zero momentum component in the interaction term, i.e. show that

$$\frac{1}{2}\int d^d r\, d^d r' \left[\hat{\rho}(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\rho}(\mathbf{r}') \;-\; \hat{\rho}(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\overline{\rho}\right] \;=\; \frac{1}{2L^d}\sum_{\mathbf{q}\neq\mathbf{0}}\hat{\rho}_{-\mathbf{q}}V(\mathbf{q})\hat{\rho}_{\mathbf{q}}$$

Here, $\overline{\rho} = \langle\hat{\rho}(\mathbf{r})\rangle$ is the average charge density of electrons, and the Fourier transforms of the density operator and interaction potential are defined as $\hat{\rho}_{\mathbf{q}} = \int d^d r\, \hat{\rho}(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}}$ and $V(\mathbf{q}) = \int d^d r\, V(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}}$, respectively. All spatial integrals are over a hypercube of volume L^d, and you may assume periodic boundary conditions. Hint: you may use that $\frac{1}{L^d}\int d^d r\, \hat{\rho}(\mathbf{r}) \approx \overline{\rho}$.

11. Screening in the Thomas Fermi approximation *4+4+4 Bonus Punkte*

Imagine placing a test charge into a jellium electron system. At first the potential due to the added test charge extends its influence to the far reaches of the system, dying slowly off as $1/r$. However, mobile electrons nearby rapidly react to the test charge, and the motions they make in response have the effect of almost completely canceling out its electric field, except within a characteristic distance called the screening length. The phenomenon can be studied in the context of Thomas-Fermi theory. The Thomas-Fermi theory allows to determine the local electron density from the following equation

$$\frac{\hbar^2}{2m}\left(3\pi^2\right)^{2/3} n^{2/3}(\mathbf{r}) \;+\; U(\mathbf{r}) \;+\; \int d^3 r' \,\frac{e^2 n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \;=\mu \quad .$$

The equation can be understood by realizing that for a homogenous electron system the first term is the kinetic energy for particles at the Fermi level, and that at zero temperature the sum of kinetic energy, potential energy, and interaction energy for particles at the Fermi level has to be equal to the chemical potential.

a) Show that the first term is indeed the kinetic energy for a pariticle at the Fermi level, i.e. calculate the kintetic energy $\epsilon(k_F)$ and express k_F in terms of the density.

b) Suppose that n_0 is the solution of this equation when the potential $U(\mathbf{r})$ of the positive background neutralizes the Coulomb potential due to the average electron density n_0 as discussed in the previous problem. If now a small potential $\delta U(\mathbf{r})$ is added, find the equation governing deviations $\delta n(\mathbf{r})$ of the density from perfect uniformity to first order in δU .

c) Consider adding one extra electron to the uniform electron gas and therefore specialize to the case $U(\mathbf{r}) = e^2/[\mathbf{r}|$. Solve the linearized equation for $\delta n(\mathbf{r})$ by Fourier transforming. The answer should be of the form $\delta n(\mathbf{r}) \sim e^{-|\mathbf{r}|/\xi}$. Identify the screening length ξ, and calculate the effective potential

$$V_{\text{eff}}(\mathbf{r}) = U(\mathbf{r}) + \int d^3r' \, \frac{e^2 \delta n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \quad .$$