
Quantum Field Theory of Many-Particle Systems - Problem Set 8

Winter Term 2011/12

Internet: You can download this problem set at <http://www.uni-leipzig.de/~rosenow>.

12. The Cooper Problem

5+5+5 Punkte

Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function

$$(1) \quad \phi(\mathbf{r} - \mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} \chi(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} .$$

In the momentum representation the Schroedinger equation has the form

$$(2) \quad \left(E - 2\frac{\hbar^2 k^2}{2m} \right) \chi(\mathbf{k}) = \int \frac{d^3k'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') \chi(\mathbf{k}') .$$

We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle: electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

$$(3) \quad \chi(\mathbf{k}) = 0 , \quad k < k_F .$$

We take the interaction of the pair to have the simple attractive form

$$(4) \quad \begin{aligned} V(\mathbf{k}_1, \mathbf{k}_2) &\equiv -g , \quad \epsilon_F \leq \frac{\hbar^2 k_i^2}{2m} \leq \epsilon_F + \hbar\omega_D , \quad i = 1, 2 ; \\ &= 0 , \quad \text{otherwise} , \end{aligned}$$

and look for a bound-state solution to the Schroedinger equation (2) consistent with the constraint (3). Since we are considering only one-electron levels which in the absence of the attraction have energies in excess of $2\epsilon_F$, a bound state will be one with energy E less than $2\epsilon_F$, and the binding energy will be

$$(5) \quad \Delta = 2\epsilon_F - E .$$

a) Show tht a bound state of energy E exists provided that

$$(6) \quad 1 = g \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} d\epsilon \frac{\rho(\epsilon)}{2\epsilon - E} ,$$

where $\rho(\epsilon)$ is the density of one-electron levels per unit volume for a given spin.

- b) Show that Eq. (6) has a solution with $E < 2\epsilon_F$ for arbitrarily weak g , provided that $\rho(\epsilon_F) \neq 0$ and that $\rho(\epsilon)$ is continuous. (Note the crucial role played by the exclusion principle: if the lower cutoff were not ϵ_F , but 0, then since $\rho(0) = 0$, there would *not* be a solution for arbitrarily weak coupling).
- c) Assuming that $\rho(\epsilon)$ differs negligibly from $\rho(\epsilon_F) \equiv \rho_F$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega_D$, show that the binding energy is given by

$$(7) \quad \Delta = 2\hbar\omega_D \frac{e^{-\frac{2}{g\rho_F}}}{1 - e^{-\frac{2}{g\rho_F}}} ,$$

or, in the weak coupling limit:

$$(8) \quad \Delta = 2\hbar\omega_D e^{-\frac{2}{g\rho_F}} .$$