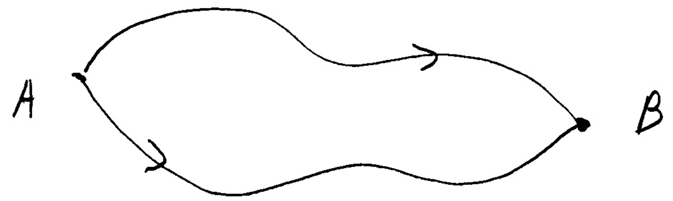
## 4.2 Interference corrections to the conductivity

Under the condition $k_F l \gg 1$ electrons can be treated as "classical particles" and the conductivity can be calculated by solving the Boltzman equation.
A full calculation of the conductivity can be performed by using methods of quantum field theory. However, the order of magnitude of these corrections can be estimated in a semiclassical framework.
Suppose that an electron, while being scattered from impurities, moves from point A to B.



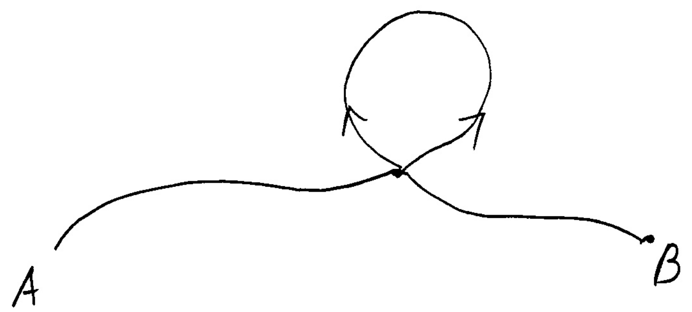
According to the principles of quantum mechanics, the amplitudes $A_i$ of different paths must be added, and the probability for going from A to B is

$$W = \left|\sum_i A_i\right|^2 = \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_j^*$$

The first term is the sum of probabilities, while the second term describes the interference of different amplitudes.

For most trajectories, interference is not important, since the length of trajectories differs by more than the Fermi wave length and therefor their relative phases are essentially random. As a consequence, the sum over these trajectories in the interference term is a sum over random phases and vanishes.

However, there are special trajectories for which the interference contribution cannot be neglected. For self-intersecting trajectories, particles travel along a closed loop. Each such trajectory corresponds to two amplitudes which differ in the direction of traversal of the loop.




The phase acquired along a given trajectory is given by

$$\Delta \varphi = \hbar^{-1} \int_A^B \underline{p} \cdot d\underline{l}$$

As with a change in the direction of motion $\underline{p}$ is replaced by $-\underline{p}$ and $d\underline{l}$ by $-d\underline{l}$, the phase $\Delta \varphi$ is the same for both trajectories. Hence the two trajectories have the same phase and we find for them

$$|A_1 + A_2|^2 = |A_1|^2 + |A_2|^2 + A_1 A_2^* + A_2 A_1^* = 4\,|A_1|^2$$

The probability for these trajectories is twice as large as the classical probability. This indicates an increase in the total scattering probability, i.e. an increase in resistivity or decrease in conductivity.

We estimate the order of magnitude of interference corrections from the enhancement of the return probability to the origin. Each classical trajectory returning ~~to~~ to the origin after time $t$ has a time reversed partner such that the QM return probability is twice the classical one. The density distribution for a particle diffusing in $d$ spatial dimensions is

$$n(\underline{r}) = \frac{1}{(4\pi D t)^{d/2}}\, e^{-|\underline{r}|^2/4Dt} .$$

Next, we calculate the volume enclosed by a given trajectory. Classically, the enclosed volume would be zero, and the probability for self-intersection would be zero as well. Quantum mechanically, the position can be determined only up to an uncertainty $\lambda_F$, and the cross section of a trajectory is $\lambda_F^{d-1}$. In order for a closed loop to occur, it is required that the final point of the trajectory enters the volume element $\lambda_F^{d-1} v_F dt$ in which the initial point lies. The probability for this to occur is

$$w(t) = \frac{\lambda_F^{d-1} v_F dt}{(4\pi t D)^{d/2}}$$

The increased return probability to the origin due to time reversed paths is determined by the sum over closed loops as

$$\frac{\Delta\sigma}{\sigma} \sim - \int_{\tau}^{\tau_\varphi} \frac{\lambda_F^{d-1} v_F dt}{(4\pi D t)^{d/2}}$$

The lower cutoff is the elastic scattering time $\tau$, the upper cutoff is the phase breaking time (decoherence time) $\tau_\varphi$. At low temperatures, dephasing due to ee-interactions is dominant and gives rise to a $\tau_\varphi \sim T^{-p}$, which grows when the temperature is lowered.

Upon integration, we find

$$\frac{\Delta\sigma}{\sigma} \sim - \frac{v_F \lambda_F^{d-1}}{(4\pi D)^{d/2}} \frac{2}{2-d} \left[ \tau_\varphi^{\frac{2-d}{2}} - \tau^{\frac{2-d}{2}} \right]$$

$$\sim - \frac{v_F \tau \lambda_F^{d-1}}{(4\pi D \tau)^{d/2}} \frac{2}{2-d} \left[ \left(\frac{\tau_\varphi}{\tau}\right)^{\frac{2-d}{2}} - 1 \right]$$

Using $l = v_F \tau$, $D \sim v_F l$, $l_\varphi = \sqrt{D\tau_\varphi}$

one obtains (neglecting factors of order unity)

$$\frac{\Delta\sigma}{\sigma} \sim \left(\frac{\lambda_F}{l}\right)^{d-1} \frac{2}{2-d}\left[\left(\frac{l_\varphi}{l}\right)^{2-d} - 1\right]$$

$d=3$: for $l_\varphi \gg l$ one has

$$\frac{\Delta\sigma}{\sigma} \sim -\frac{\lambda_F^2}{l^2}$$

There is a localization transition for a finite value

$\frac{\lambda_F}{l} \approx 1$.

$d=2$: $$\frac{\Delta\sigma}{\sigma} \sim -\frac{\lambda_F}{l}\ln\frac{l_\varphi}{l}$$

The conductivity is reduced logarithmically in $l_\varphi$.
At $T=0$, all states are localized. The localization length can be estimated from the condition

$\frac{\Delta\sigma(\xi_{loc})}{\sigma} = 1 \Rightarrow \xi_{loc} = l\, e^{l/\lambda_F}$. Using

the expression $\frac{\sigma_\square}{\left(\frac{e^2}{h}\right)} \equiv g_\square \approx \frac{l}{\lambda_F}$, this translates

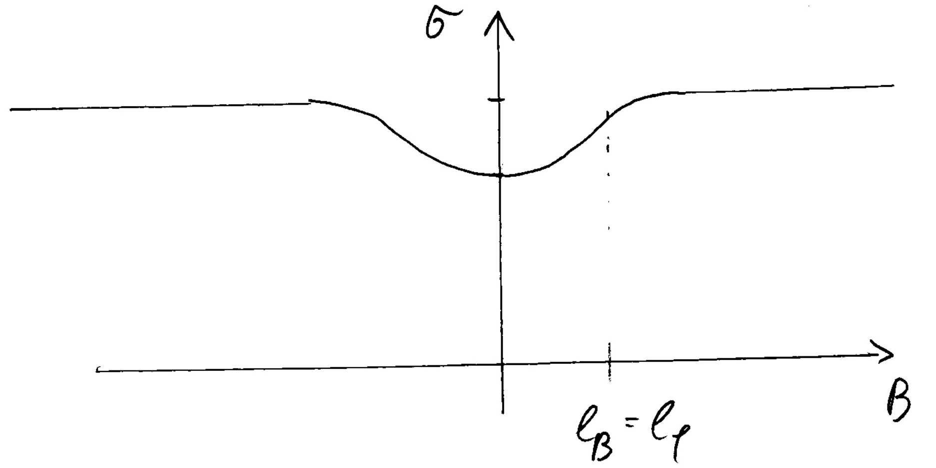
into $\xi_{loc} \approx l\, e^{g_\square}$

$d=1$: $\quad \frac{\Delta\sigma}{\sigma} \sim \frac{l_\varphi}{l}$

Conductivity is reduced as a power of temperature.
All states are localized on a scale $\xi_{loc} \sim l$.

A magnetic field destroys the symmetry between time reversed paths and hence weak localization effects. For $l_B < l_\varphi$ with $l_B = \sqrt{\frac{\hbar}{eB}}$ denoting the magnetic length, $\sigma$ returns to its classical value

$\rightarrow$ measurement of $l_\varphi$ is possible!



# Dephasing of diffusive electrons

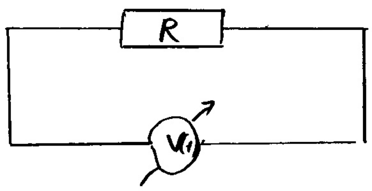
Consider the time evolution of an electron in an electromagnetic potential $H = e\,V(t)$

Wave function $\psi(x)$ acquires time dependence

$$\psi(x,t) = e^{-\frac{ie}{\hbar}\int_0^t dt'\, V(t')}\, \psi(x,0)$$

We imagine that the potential $V(t)$ is due to the motion of all other electrons in the system. Since this problem is too complicated to solve, we describe $V(t)$ as a random variable with expection value $\langle V(t)\rangle = 0$ and variance $\langle V(t)\,V(0)\rangle = 4\,k_B T\,R\,\delta(t)$

in analogy to the thermal Nyquist-Johnson noise across a resistor




We will determine the value of $R$ in a moment.

Now $\langle \psi(x,t)\rangle = \psi(x,0)\,\left\langle e^{-\frac{ie}{\hbar}\int_0^t dt'\, V(t')}\right\rangle$

$$= \psi(x,0)\, e^{-\frac{1}{2}\left(\frac{e}{\hbar}\right)^2 \int_0^t dt' \int_0^t dt'' \langle V(t')V(t'')\rangle}$$

valid for Gaussian random variables

$$\Rightarrow \quad \langle \Psi(x,t) \rangle = \Psi(x,0)\, e^{-\frac{1}{2}\frac{e^2}{\hbar^2} 4 k_B T R t}$$

Due to the fluctuations in the external potential, the expectation value of the wave function (i.e. the expectation value for coherent propagation of a particle) decreases exponentially in time. We define a phase breaking time or decoherence time $\tau_\varphi$ via $\langle \Psi(x,t) \rangle = \Psi(x,0)\, e^{-t/\tau_\varphi}$ and find by comparison

$$\frac{1}{\tau_\varphi} = \frac{2e^2}{\hbar^2} R k_B T$$

To determine R, we use the fact that electrons in a disordered metal or semiconductor perform a diffusive motion, such that

$$\langle [x(t) - x(0)]^2 \rangle = 2d\,Dt \quad , \quad \text{where } d$$

denotes the spatial dimension, and $D = \frac{1}{3} \ell v_F$ the diffusion constant ($\ell$ = elastic mean free path, $v_F$ Fermi velocity). The typical distance $L$ travelled by a particle in time $t$ is hence $L = \sqrt{2dDt}$. As value for R we choose the resistance of a hypercube of size $L$

Conductance $G(L) = \sigma L^{d-2} \Rightarrow R(L) = \frac{1}{G(L)} = \frac{1}{\sigma} L^{2-d}$

$$\Rightarrow \quad \frac{1}{\tau_\varphi} = \frac{2e^2}{\hbar^2} k_B T \frac{L^{2-d}}{\sigma}$$

To find a close form expresion for $\frac{1}{\tau_\varphi}$, we chose $L = l_\varphi$ with $l_\varphi = \sqrt{2dDt}$ as

$$\Rightarrow \quad \frac{1}{\tau_\varphi} = \frac{2e^2 k_B T}{\hbar^2 \sigma} \left(2dD\tau_\varphi\right)^{\frac{2-d}{2}}$$

$$\Rightarrow \quad \tau_\varphi^{-1-1+\frac{d}{2}} = \frac{2e^2 k_B T}{\hbar^2 \sigma} \left(2dD\right)^{\frac{2-d}{2}}$$

$$\Rightarrow \quad \tau_\varphi^{-1} = \left[ \frac{2e^2 k_B T}{\hbar^2 \sigma} \left(2dD\right)^{\frac{2-d}{2}} \right]^{\frac{2}{4-d}}$$

$$\tau_\varphi^{-1} \propto \begin{cases} (k_B T)^2 & \text{in } 3d \\ k_B T & \text{in } 2d \\ (k_B T)^{\frac{2}{3}} & \text{in multi-channel 1d wire} \end{cases}$$