Quantum Field Theory of Many-Particle Systems - Problem Set 2

Summer Term 2014

Due Date: Thursday, April 24, 3:15 p.m., tutorial session

Internet: You can download this problem set at http://www.uni-leipzig.de/∼rosenow.

3. Quantization of the Radiation Field   *5+5+5 Punkte*

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2} \int_\Omega d^3x \left[\epsilon_0 \left(\partial_t \mathbf{A} \right)^2 + \frac{1}{\mu_0} \mathbf{A} \nabla^2 \mathbf{A} \right] \ .$$

Here, ϵ_0 denotes the vacuum dielectric constant, μ_0 the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

a) Derive the equation of motion for $\mathbf{A}$ by extremizing the action $S = \int dt L$.

b) Find eigenfunctions $\mathbf{A_k}$ and eigenvalues $\omega_\mathbf{k}^2$ of the equation

$$-\nabla^2 \mathbf{A} = \frac{\omega_\mathbf{k}^2}{c^2} \mathbf{A}$$

by using periodic boundary conditions. It may be useful to introduce for each $\mathbf{k}$ a set of orthonormal vectors $\{\xi_{\mathbf{k},1}, \xi_{\mathbf{k},2}\}$ which are both perpendicular to $\mathbf{k}$. The time-dependent solution has then a series expansion

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{\Omega} \sum_{\mathbf{k},i} \alpha_{\mathbf{k},i}(t) \mathbf{A}_{\mathbf{k},i}(\mathbf{x}) \ .$$

Insert this series expansion into the Lagrangian, and find the momenta

$$\pi_{\mathbf{k},i} = \frac{\partial L}{\partial \dot{\alpha}_{\mathbf{k},i}} \ .$$

canonically conjugate to the coordinates $\alpha_{\mathbf{k},i}$. Use the Legendre transform $H = \sum_{\mathbf{k},i} \pi_{\mathbf{k},i} \dot{\alpha}_{\mathbf{k},i} - L(\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i})$ to obtain the Hamiltonian.

c) The classical Hamiltonian $H(\{\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i}\})$ can be quantized by imposing canonical commutation relations

$$[\alpha_{\mathbf{k},i}, \alpha_{\mathbf{q},j}] = 0, \quad [\pi_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = 0 , \quad [\alpha_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = i\hbar \, \delta_{\mathbf{k},\mathbf{q}} \delta_{i,j} \ .$$

on coordinates $\alpha_{\mathbf{k},i}$ and canonically conjugate momenta $\pi_{\mathbf{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0 \omega_\mathbf{k}}{2\hbar}} \left(\alpha_{\mathbf{k},j} + \frac{i}{\epsilon_0 \omega_\mathbf{k}} \pi_{-\mathbf{k},j} \right) \ , \quad a_{\mathbf{k},j}^\dagger = \sqrt{\frac{\epsilon_0 \omega_\mathbf{k}}{2\hbar}} \left(\alpha_{-\mathbf{k},j} - \frac{i}{\epsilon_0 \omega_\mathbf{k}} \pi_{\mathbf{k},j} \right)$$

Show that the $a_{\mathbf{k},j}$, $a_{\mathbf{k},j}^\dagger$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of the $a_{\mathbf{k},j}$, $a_{\mathbf{k},j}^\dagger$.