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Quantum Field Theory of Many-Particle Systems - Problem Set 3

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Internet: You can download this problem set at http://www.uni-leipzig.de/∼stp.

4. Quantum Harmonic Oscillator

5+5+5 Punkte

The harmonic oscillator provides a valuable environment in which the path integral method can be explored. It is one of the few examples, for which the path integral can be evaluated exactly. Its Hamiltonian is given by

$$\hat{H} \;=\; \frac{\hat{p}^2}{2m} \;+\; \frac{1}{2}m\omega^2\hat{x}^2 \quad .$$

Use the formula

$$\text{(1)} \qquad \langle x_F|e^{-i\hat{H}t/\hbar}|x_i\rangle \;=\; e^{\frac{i}{\hbar}S[x_{cl}]}\int_{r(0)=0,r(t)=0} D\,[r]\,e^{\frac{i}{2\hbar}\int_0^t dt'\,r(t')\left[-m\partial_{t'}^2 \;-\; m\omega^2\right]r(t')}$$

to evaluate the propagator for the quantum harmonic oscillator.

a) Find a solution to the classical equation of motion which satisfies the boundary conditions $x_{cl}(0) = x_I$ and $x_{cl}(t) = x_F$. Use this solution to evaluate the classical action and the exponential prefactor in Eq. (1).

b) Find a Fourier representation for $r(t)$ which satisfies the boundary conditions $r(0) = 0$ and $r(t) = 0$. Use this Fourier representation to determine the eigenvalues λ_n^A of the differential operator $\hat{A}$ in the exponent of the path integral.

c) In order to evaluate the determinant as the products of eigenvalues of $\hat{A}$, use the result for a free particle. We denote the differential operator for the free particle by $\hat{B}$. The free particle fluctuation factor is

$$F_B \;=\; \sqrt{\frac{m}{it2\pi\hbar}} \quad .$$

Use the relation

$$F_A = F_B\prod_{n=1}^{\infty}\left(\frac{\lambda_n^A}{\lambda_n^B}\right)^{-1/2}$$

together with the mathematical identity $z/\sin z \;=\; \prod_{n=1}^{\infty}(1-z^2/\pi^2n^2)^{-1}$ to compute the propagator for the quantum harmonic oscillator.