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Quantum Field Theory of Many-Particle Systems - Problem Set 8

Summer Term 2014

Internet: You can download this problem set at http://www.uni-leipzig.de/∼rosenow.

12. The Cooper Problem

5+5+5 Punkte

Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function

(1)
$$\phi(\mathbf{r}-\mathbf{r}') \; = \; \int \frac{d^3k}{(2\pi)^3}\,\chi(\mathbf{k})\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \quad .$$

In the momentum representation the Schroedinger equation has the form

(2)
$$\left(E - 2\frac{\hbar^2 k^2}{2m}\right)\chi(\mathbf{k}) \; = \; \int \frac{d^3k}{(2\pi)^3}\; V(\mathbf{k},\mathbf{k}')\,\chi(\mathbf{k}') \quad .$$

We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle: electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

(3)
$$\chi(\mathbf{k}) \; = \; 0 \; , \quad k < k_F \quad .$$

We take the interaction of the pair to have the simple attractive form

(4)
$$\begin{aligned} V(\mathbf{k_1},\mathbf{k_2}) \; &\equiv \; -g \; , \quad \epsilon_F \le \frac{\hbar^2 k_i^2}{2m} \le \epsilon_F + \hbar\omega_D \; , \quad i = 1,2 \; ; \\ &= \; 0 \; , \qquad \text{otherwise} \; , \end{aligned}$$

and look for a bound-state solution to the Schroedinger equation (2) consistent with the constraint (3). Since we are considering only one-electron levels which in the absence of the attraction have energies in excess of $2\epsilon_F$, a bound state will be one with energy E less than $2\epsilon_F$, and the binding energy will be

(5)
$$\Delta \; = \; 2\epsilon_F - E \quad .$$

a) Show tht a bound state of energy E exists provided that

(6)
$$1 \; = \; g \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} d\epsilon \; \frac{\rho(\epsilon)}{2\epsilon - E} \; ,$$

where $\rho(\epsilon)$ is the density of one-electron levels per unit volume for a given spin.

b) Show that Eq. (6) has a solution with $E < 2\epsilon_F$ for arbitrarily weak g, provided that $\rho(\epsilon_F) \neq 0$ and that $\rho(\epsilon)$ is continuous. (Note the crucial role played by the exclusion principle: if the lower cutoff were not ϵ_F, but 0, then since $\rho(0) = 0$, there would *not* be a solution for arbitrarily weak coupling).

c) Assuming that $\rho(\epsilon)$ differs negligibly from $\rho(\epsilon_F) \equiv \rho_F$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega_D$, show that the binding energy is given by

$$\Delta = 2\hbar\omega_D \frac{e^{-\frac{2}{g\rho_F}}}{1 - e^{-\frac{2}{g\rho_F}}} \quad , \tag{7}$$

or, in the weak coupling limit:

$$\Delta = 2\hbar\omega_D \, e^{-\frac{2}{g\rho_F}} \quad . \tag{8}$$