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Quantum Field Theory of Many-Particle Systems - Problem Set 9

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Internet: You can download this problem set at http://www.uni-leipzig.de/∼rosenow.

13. Electron Tunneling between two metals *4+4+4+8+4+4 Punkte*

The most important verifications of the BCS theory of superconductivity came from electron tunneling experimetns, in which the energy gap as a function of temperature was measured and showed excellent agreement with the BCS theory. In this problem set electron tunneling between two normal metals will be discussed. In next week's problem set tunneling between normal metal and superconductor, and between two superconductors will be explored. Please note that you can solve most parts of the problem set independently of the other parts. For the most difficult part d) you get bonus points.

We consider two metallic regions, one on the left (L) and one on the right (R), which are separated by an insulating barrier. This setup is described by the Hamiltonian:

$$\begin{aligned} H &= H_R + H_L + H_T \\ H_T &= \sum_{\mathbf{k},\mathbf{p}} \left(T_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{k}} c_{\mathbf{p}} + h.c. \right) \quad . \end{aligned}$$

The first term H_R is the Hamiltonian for particles on the right side of the tunneling junction. It contains all many-body interactions. For a normal metal, it is characterized by an energy function $\xi_{\mathbf{p}}$ and a density of states ρ_F. Similarly, H_L describes all the physics for particles on the left side of the junction. These two parts of the Hamiltonian are considered to be strictly independent. Not only do they commute, $[H_L, H_R] = 0$, but they commute term by term. The Hamiltonian on the right can be expressed in terms of one set of operators $c_{\mathbf{k}}$, and the Hamiltonian on the left side by another set $c_{\mathbf{p}}$. These two sets of fermionic operators are independent, i.e.

$$\{c_{\mathbf{k}}, c^\dagger_{\mathbf{p}}\} = 0 \quad .$$

As in the end we will be interested in tunnel voltages of the order of the Debye energy, it is a good approximation to assume that the tunnel matrix elements are independent of energy

$$T_{\mathbf{k},\mathbf{p}} \equiv T_0 \quad .$$

a) The tunneling current through the insulating region is expressed as the rate of change of the number of particles on, for example, the left-hand side of the junction $N_L = \sum_{\mathbf{p}} c^\dagger_{\mathbf{p}} c_{\mathbf{p}}$. Show that this rate of change

$$\dot{N}_L = i\,[H, N_L] = i[H_T, N_L]$$

is given by

$$\dot{N}_L = i \sum_{\mathbf{k},\mathbf{p}} \left(T_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{k}} c_{\mathbf{p}} - T^\star_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{p}} c_{\mathbf{k}} \right) \quad .$$

b) We consider $H_0 = H_L + H_R$ as the unperturbed Hamiltonian, and H_T as the perturbation. Then, we can use the theory of linear response to obtain the tunnel current $I(t) = -e\langle \dot{N}_L(t)\rangle$ as

$$I(t) = -ei \int_{-\infty}^{t} dt' \langle [\dot{N}_L(t), H_T(t')] \quad .$$

Here, the time dependence of operators is due to H_0, and we assume that there exists a difference in chemical potential $V = \mu_L - \mu_R$. Show that the current due to tunneling of single electrons is given by (i.e. assume that expectation values of the type $\langle c_{\mathbf{k}} c_{\mathbf{k}'} c^\dagger_{\mathbf{p}} c^\dagger_{\mathbf{p}'}\rangle = 0$ vanish)

$$I_{\text{single}} = e \int_{-\infty}^{\infty} dt' \Theta(t-t') \left\{ e^{-ieV(t'-t)} \langle [A(t), A^\dagger(t')] \rangle - e^{-ieV(t-t')} \langle [A^\dagger(t), A(t')] \rangle \right\} \quad .$$

Here, the operator A is defined as

$$A = \sum_{\mathbf{k},\mathbf{p}} T_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{k}} c_{\mathbf{p}} \quad .$$

c) Introducing the retarded correlation function

$$C^{+}_{A,A^\dagger}(t) = -i\Theta(t)\langle [A(t), A^\dagger(0)] \rangle \quad ,$$

show that the single particle tunnel current can be expressed in terms of its Fourier transform

$$C^{+}_{A,A^\dagger}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} C^{+}_{A,A^\dagger}(t)$$

as

$$I_{\text{single}} = ie \left[C^{+}_{A,A^\dagger}(eV) - \left(C^{+}_{A,A^\dagger}(eV) \right)^{\star} \right]$$

$$= -2e \, \text{Im}[C^{+}_{A,A^\dagger}(-eV)] \quad .$$

d) For this part of the problem you get bonus points. Show that

$$C^{\tau}_{A,A^\dagger}(i\omega_n) = \sum_{\mathbf{k},\mathbf{p}} |T_{\mathbf{k},\mathbf{p}}|^2 \frac{n_F(\xi_{\mathbf{k}}) - n_F(\xi_{\mathbf{p}})}{i\omega_n + \xi_{\mathbf{k}} - \xi_{\mathbf{p}}} \quad .$$

e) Show that by analytic continuation $i\omega_n \to \omega + i\eta$ of the result d) the tunnel current is

$$I_{\text{single}} = 4\pi e \sum_{\mathbf{k},\mathbf{p}} |T_{\mathbf{k},\mathbf{p}}|^2 \, \delta(eV + \xi_{\mathbf{k}} - \xi_{\mathbf{p}}) \, [n_F(\xi_{\mathbf{k}}) - n_F(\xi_{\mathbf{p}})] \quad .$$

f) Now make the assumption that $T_{\mathbf{k},\mathbf{p}} \equiv T_0$. Introduce the total densities of states $N_{R,L} = L^d \rho_{R,L}(\epsilon_F)$ which are densities of states per volume multiplied by the respective system volumes to convert the momentum sums to integrals, and show that the tunnel current is given by

$$I_{\text{single}} = 4\pi e^2 N_R N_L \, |T_0|^2 \, V \quad .$$