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Quantum Physics of Nanostructures - Problem Set 2

Winter term 2014/2015

Due date: The problem set will be discussed Wednesday, 05.11.2014.

Internet: Course information and problem sets are available online at http://www.uni-leipzig.de/~stp/QP_of_Nanostructures_WS1415.html.

4. Unitarity and Time-Reversal Symmetry I

3+3+3 Points

In the following, consider the single-channel scattering matrix S that maps the amplitudes of incoming states (i_L, i_R, where L: left, R: right) to amplitudes of outgoing states (o_L, o_R) of some scattering region,

$$\begin{pmatrix} o_L \\ o_R \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} i_L \\ i_R \end{pmatrix}, \text{ where } S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}.$$

In the absence of magnetic fields or magnetic impurities, the Hamiltonian entering the Schrödinger equation obeys *time-reversal symmetry* (TRS): this tells us, that under $t \to -t$, for every solution ψ of the Schrödinger equation, ψ^* is a solution to the time-reversed equation (for simplicity, the spin degree of freedom is not considered here). In the scattering-matrix formalism, besides complex-conjugating amplitudes, incoming states become outgoing ones under time reversal and vice versa, while scattering is described by the same S, i.e., the following relation holds:

$$\begin{pmatrix} i_L^* \\ i_R^* \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} o_L^* \\ o_R^* \end{pmatrix}.$$

(a) From unitarity ($S^\dagger S = \mathbb{1}$) of the scattering matrix, derive the relations

$$T + R' = 1, \; T' + R = 1, \; T + R = 1 \text{ and } \frac{r}{t'} = -\left(\frac{r'}{t}\right)^*,$$

where $|r|^2 = R$, $|t|^2 = T$ and $|r'|^2 = R'$, $|t'|^2 = T'$.

(b) Show, that TRS implies $S = S^T$, where S^T denotes the transpose of S.

(c) Additionally assuming TRS, derive the following relation for amplitudes:

$$\frac{r}{t} = -\left(\frac{r'}{t}\right)^*.$$

5. Unitarity and Time-Reversal Symmetry II

2+2+2 Points

Check whether the following 2×2 matrices qualify as a single-channel scattering matrix and if so, whether the scattering is symmetric under time reversal:

(a) $M_1 = \exp(\mathrm{i}\sigma_x \alpha)$, $\alpha \in [0, 2\pi)$,

(b) $M_2 = \exp(\mathrm{i}\sigma_y \beta)$, $\beta \in [0, 2\pi)$.

(c) $M_3 = \exp(\sigma_z \gamma)$, $\gamma \in [0, 2\pi)$.

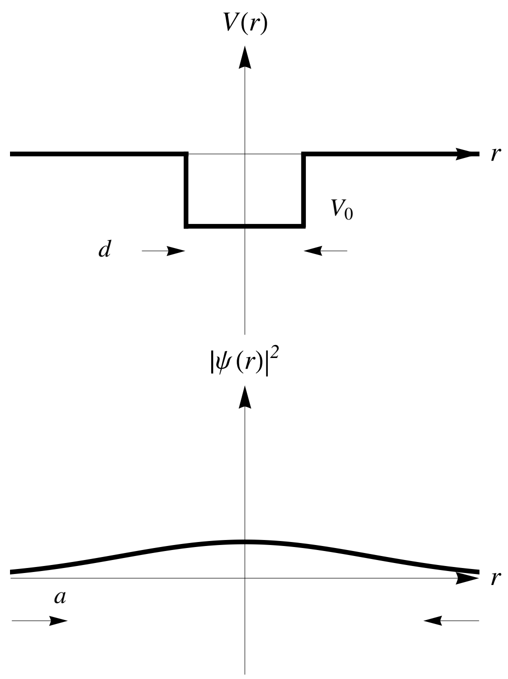
Here, σ_x, σ_y and σ_z denote the Pauli matrices and are explicitly given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

6. Weakly Bound State

5 Points

A particle moves in a D-dimensional rotationally symmetric potential well of width d and depth V_0 (see Figure). Assume that the width of the wave function a is much larger than the width of the potential well d, such that $a \gg d$, as illustrated in the Figure. Your task will be to analyze, whether or not an arbitrarily shallow ($V_0 \to 0^-$) attractive potential well in dimension D can bind a particle.



Use the Heisenberg uncertainty relation for position and momentum to estimate the kinetic and potential energy of the ground state. Minimize the energy function $E(a)$ you get from this estimate with respect to the width of the wave function and obtain the minimizing a_0. For which dimensions D does a bound state ($E(a_0) < 0$) always exist?