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Quantum Physics of Nanostructures - Problem Set 3

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Due date: The problem set will be discussed Wednesday, 26.11.2014.

Internet: Course information and problem sets are available online at http://www.uni-leipzig.de/∼stp/QP_of_Nanostructures_WS1415.html.

7. Current Noise

5+5+5+5 Points

In this problem, you will derive an expression for the zero-frequency current noise of an electrical circuit. To achieve this, we start from an analog system, describing the motion of a classical particle under influence of friction and a fluctuating force. This mechanical analog model is governed by a stochastic differential equation, a so-called Langevin equation, of the form

$$m\dot{v}(t) = -\eta v(t) + F(t),$$

written in terms of the particle's velocity $v(t)$ at time t. The mass of the particle is denoted by m, η is a friction coefficient, and $F(t)$ denotes a fluctuating random force acting on the particle. We assume the force $F(t)$ to be characterized by a white noise correlation function, $\langle F(t)F(t')\rangle = a\delta(t-t')$ with a constant a, where the average is taken with respect to the distribution of $F(t)$.

(a) Make an ansatz $v(t) = v_\omega \mathrm{e}^{-\mathrm{i}\omega t}$, $F(t) = F_\omega \mathrm{e}^{-\mathrm{i}\omega t}$ for the time dependence of velocity and force. Derive an algebraic relation between v_ω and F_ω.

Remark: Since the equation of motion is linear, the general solution can be obtained by integrating over ω, i.e.,

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \mathrm{e}^{-\mathrm{i}\omega t}\, v_\omega.$$

(b) From this algebraic relation, derive a relation between $\langle v_\omega v_{\omega'}\rangle$ and $\langle F_\omega F_{\omega'}\rangle$ and express the right-hand side through the inverse Fourier transform $F_\omega = \int_{-\infty}^{+\infty} dt\, \mathrm{e}^{\mathrm{i}\omega t}\, F(t)$.

(c) From the result in part (b), derive an expression for the spectral density S^v_ω of the velocity-velocity correlator, defined through

$$\langle v_\omega v_{\omega'}\rangle = 2\pi S^v_\omega \delta(\omega+\omega').$$

To determine a, employ the Fluctuation-Dissipation-Theorem (FDT) for the real part of the mobility for $\omega \to 0$,

$$\mathrm{Re}[\mu_\omega] = \frac{1}{2\hbar\omega}\, S^v_\omega \left(1 - \mathrm{e}^{-\hbar\omega/k_B T}\right).$$

The mobility μ_ω is defined as the velocity to force ratio.

Hint: The delta function has the property $\int_{-\infty}^{+\infty} d\omega'\, \delta(\omega-\omega') f(\omega') = f(\omega)$ and can be represented as $\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dt\, \mathrm{e}^{\mathrm{i}(\omega-\omega')t}$.

You are now in a position to derive an expression for the zero-frequency current noise (Nyquist-Johnson noise)

$$\lim_{\omega\to 0} S^I_\omega = \lim_{\omega\to 0} \int_{-\infty}^{+\infty} dt\, \mathrm{e}^{\mathrm{i}\omega t} \langle I(t) I(0) \rangle,$$

by exploiting the analogy of the particle motion to the dynamics of the current $I(t)$ in an electrical circuit, characterized by an inductance L, a resistance R, and voltage fluctuations $V(t)$:

$$L\dot{I}(t) = -RI(t) + V(t).$$

The results from above carry over to the situation of the electrical circuit by making the replacements $m \to L$, $\eta \to R$ and $v \to I$, $F \to V$.

(d) Derive an expression for the current noise S^I_ω in terms of the conductance of the circuit and consider the limit $\omega \to 0$.

8. Einstein Relation

0 Points

In this bonus problem, you will derive the Einstein relation between diffusion coefficient and mobility, building on results from above.

(a) Fourier transforming the velocity-velocity correlator $\langle v_\omega v_{\omega'} \rangle$ in the frequency domain, one obtains the velocity-velocity correlator in the time domain as

$$\langle v(t) v(0) \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{a}{\eta^2 + m^2\omega^2} \mathrm{e}^{-\mathrm{i}\omega t} = \frac{a}{2m\eta} \mathrm{e}^{-\frac{\eta}{m}|t|}.$$

Convince yourself, that in the limit of large η, it is a reasonable approximation to treat $\frac{\eta}{2m} \mathrm{e}^{-\frac{\eta}{m}|t|}$ as a delta function in time.

(b) Use the representation $x(t) - x(0) = \int_0^t dt'\, v(t')$ for the particle's position at time t to obtain an expression for the mean-squared displacement $\langle [x(t) - x(0)]^2 \rangle$.

(c) Assuming diffusive motion, derive the Einstein relation between diffusion coefficient D and mobility μ (use your result for a found above).