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Advanced Quantum Mechanics - Problem Set 0

Winter Term 2019/20

Due Date: This problem set is discussed in the seminars on Wednesday, October 23, and Friday, October 25.

Internet: Advanced Quantum Mechanics exercises

The aim of the problem set is to get familiar with the Dirac notation.

1. Two-level system *3 Points*

Consider the Hamiltonian of a two-level system

$$\hat{H} = a\big(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\big),$$

where $a > 0$ has dimensions of an energy. Calculate the energy eigenvalues and eigenstates with respect to the orthonormal basis $\{|1\rangle, |2\rangle\}$.

2. Unitary transformation *1+2 Points*

Consider the unitary transformation $|\psi'\rangle = \hat{U}|\psi\rangle$.

(a) Show that the operator $\hat{A}$ has to be transformed as $\hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger$

(b) Show that with these definitions the following properties of the operators are conserved in the transformation:

- (i) linearity and hermiticity
- (ii) commutation relations
- (iii) the eigenvalue spectrum
- (iv) the algebraic relations $\hat{F} = \hat{K} + \hat{M}$ and $\hat{F} = \hat{K}\hat{M}$

3. Momentum representation *2+2 Points*

Let $|\alpha\rangle$ and $|\beta\rangle$ be arbitrary ket-vectors. Use the normalization $\langle p|p'\rangle = \delta(p-p')$ and completeness relation $\int dx|x\rangle\langle x| = \hat{\mathbb{1}}$ to obtain an expression for $\langle x|p\rangle$. Show then explicitly

(a) $\langle p|\hat{x}|\alpha\rangle = i\hbar\frac{\partial}{\partial p}\psi_\alpha(p)$,

(b) $\langle\beta|\hat{x}|\alpha\rangle = \int dp\ \psi^*_\beta(p)i\hbar\frac{\partial}{\partial p}\psi_\alpha(p)$.

Here $\psi_\alpha(p) \equiv \langle p|\alpha\rangle$ and $\psi_\beta(p) \equiv \langle p|\beta\rangle$ are one dimensional wave functions in momentum representation and $\hat{x}$ is the position operator.

4. Change of representation *4 Points*

Let's denote the eigenstate of the position operator $\hat{x}$ with eigenvalue x as $|x\rangle$, the eigenstate of the momentum operator $\hat{p}$ with eigenvalue p as $|p\rangle$ and the eigenstate of the Hamilton operator $\hat{H} = \frac{\hat{p}^2}{2m}$ with energy E as $|E\rangle$. Assume that the state $|\Psi\rangle$ in the momentum representation is given as $\langle p|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(-ix_0\frac{p}{\hbar})$.

(a) Calculate $\langle x|\Psi\rangle$. How can the state $|\Psi\rangle$ therefore be described?

(b) Use the eigenvalue equation for $\hat{H}$ and the matrix elements $\langle x|\hat{H}|x'\rangle = -\frac{d^2}{dx^2}\delta(x-x')\frac{\hbar^2}{2m}$ to derive a differential equation for $\Psi_E(x) = \langle x|E\rangle$ and solve $\Psi_E(x)$.

(c) Calculate the matrix elements $\langle E|\hat{H}|E'\rangle$.