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Advanced Quantum Mechanics - Problem Set 1

Winter Term 2019/20

Due Date: Hand in solutions to problems marked with * before the lecture on **Friday, 25.10.2019, 09:15**. The problem set will be discussed in the tutorials on Wednesday, 30.10.2019, and Friday, 01.11.2019

1. Simultaneous eigenstates

4+2+2 Points

(a) Let $\hat{A}$ and $\hat{B}$ be two operators which commute. Show that there exists a common set of eigenstates of the two operators. Distinguish between the case where the states are non-degenerate and n-fold degenerate.

(b) Assume now that $|\Psi\rangle$ is a simultaneous eigenstate of $\hat{A}$ and $\hat{B}$, and that $\hat{A}$ and $\hat{B}$ anti-commute: $\hat{A}\hat{B} + \hat{B}\hat{A} = 0$. What can you say about the eigenvalues of the two operators?

(c) Give a concrete example of your result from part (b) using the parity and momentum operators.

*2. Symmetric double-well potential

3+4 Points

Consider a symmetric rectangular double-well potential

$$V(x) = \begin{cases} \infty, & |x| > a + b, \\ 0, & a < |x| < a + b, \\ V_0, & |x| < a, \end{cases}$$

with $V_0 > 0$.

(a) Determine the general solution of the Schrödinger equation in the different regions, and use suitable boundary conditions to connect these solutions for subregions. Determine the quantization condition for the parameters in your solution.

Hint: Due to the symmetry of the problem you can choose solutions which are also eigenstates of parity.

(b) Assuming V_0 is very large, calculate the energies of the ground state and the first excited state from the quantization condition derived in a).

*3. Parity operator

1+2+2 Points

Consider a one-dimensional real-space wave-function $\psi(x)$ and let $\hat{P}$ denote the parity operator such that $\hat{P}\psi(x) = \psi(-x)$.

(a) Show that $\hat{P}$ commutes with the Hamiltonian $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$ as long as $V(x)$ is an even function in x, i.e. $V(x) = V(-x)$.

(b) Starting from the Rodrigues formula for Hermitian polynomials, $H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$ with $n \in \mathbb{N}$, show that the eigenfunctions $\psi_n(x)$ of the one-dimensional harmonic oscillator, with mass m and frequency ω, are also eigenfunctions of the parity operator. What are the eigenvalues?

(c) Define the operator

$$\hat{\Pi} = \exp\left[i\pi\left(\frac{1}{2\alpha}\hat{p}^2 + \frac{\alpha}{2\hbar^2}\hat{x}^2 - \frac{1}{2}\right)\right], \quad \alpha \in \mathbb{R}^+,$$

where $\hat{x}$ and $\hat{p}$ denote the position and momentum operators. Show that $\hat{\Pi}$ is a parity operator.

Hint: Consider $\hat{\Pi}\psi(x)$ and expand $\psi(x)$ with respect to a suitable basis.