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Advanced Quantum Mechanics - Problem Set 7

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 06.12.2021, 12:00**. The problem set will be discussed in the tutorials on Wednesday, 08.12.2021, and Friday, 10.12.2021

*1. Continuity equation for the Dirac equation *5 Points*

(Note that for bachelor students this problem is not mandatory in the course with the lower number of credit points.)

Prove the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0,$$

with

$$\boldsymbol{j} = \Psi^\dagger \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \Psi,$$

and $\rho = \Psi^\dagger \Psi$ for all solutions Ψ of the Dirac equation.

2. Free particle solutions of the Dirac equation *3 Points*

(Note that for bachelor students this problem is not mandatory in the course with the lower number of credit points.)

Calculate the eigenvalues of the free-particle Dirac equation

$$\begin{pmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

3. Klein Tunneling in graphene

1+3+6 Points

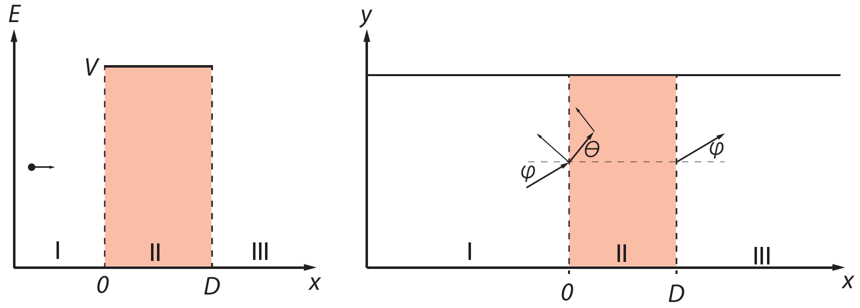

Figure 1: Left: Schematic drawing of a Dirac electron incident on a potential barrier. Right: Definition of the angles used in the problem. Assume that the sample is infinite in the y-direction.

Consider a Dirac electron with energy E incident on a potential barrier of size V as shown in the figure.

(a) Why is it sufficient to only require continuity of the wave-function and not its derivative?

(b) Assume the electron is incident at some angle ϕ in regions I and III and θ in region II, such that $k_x = k\cos\phi$, $k_y = k\sin\phi$ in regions I and III, while $\theta = \arctan(k_y/q_x)$ with $q_x = \sqrt{(V-E)^2/v^2 - k_y^2}$ and $v = |\boldsymbol{k}|/m$ in region II. Explain why the wave-functions in the different regions can be written as

$$\psi_{\mathrm{I}}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ se^{i\phi} \end{pmatrix} e^{i(k_x x + k_y y)} + \frac{r}{\sqrt{2}}\begin{pmatrix} 1 \\ se^{i(\pi-\phi)} \end{pmatrix} e^{i(k_y y - k_x x)},$$

$$\psi_{\mathrm{II}}(x) = \frac{a}{\sqrt{2}}\begin{pmatrix} 1 \\ s'e^{i\theta} \end{pmatrix} e^{i(q_x x + k_y y)} + \frac{b}{\sqrt{2}}\begin{pmatrix} 1 \\ s'e^{i(\pi-\theta)} \end{pmatrix} e^{i(k_y y - q_x x)},$$

$$\psi_{\mathrm{III}}(x) = \frac{t}{\sqrt{2}}\begin{pmatrix} 1 \\ se^{i\phi} \end{pmatrix} e^{i(k_x x + k_y y)}.$$

Here $s = \mathrm{sgn}(E)$ and $s' = \mathrm{sgn}(E - V)$. What is the physical significance of r, a, b, and t?

(c) Use the continuity of the wave-function to calculate the transmission through the barrier $T(\theta, \phi, Dq_x) = |t|^2$. What do you get for $Dq_x = n\pi$ with n integer? For general values of Dq_x, investigate what happens when $\phi, \theta \to 0$.

Hint: You might want to use a computer algebra system to solve the resulting linear equation system for t, and to compute $|t|^2$.