
Advanced Quantum Mechanics - Problem Set 8

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 13.12.2021, 12:00**.
The problem set will be discussed in the tutorials on Wednesday, 15.12.2021, and Friday, 17.12.2021

1. Commutators and products of Dirac matrices *2+2+1 Points*

(Note that for bachelor students this problem is not mandatory in the course with the lower number of credit points.)

Consider the Dirac matrices

$$\alpha = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices and $\mathbb{1}_2$ is the 2-dimensional unit matrix. Define also

$$\Sigma = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

- (a) Show that (i) $\beta\Sigma_i = \Sigma_i\beta$, and that (ii) $[\alpha_i, \Sigma_j] = 2i\epsilon_{ijk}\alpha_k$.
(b) Show that

$$\gamma^1\gamma^3 = i \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}.$$

You can use the representation of γ -matrices stated in Problem 3 of Problem Set 6.

2. Four-current for the free particle solutions of the Dirac equation

1+4 Points

The free particle solutions of the Dirac equation can be written using

$$\mathbf{u}_R^{(+)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{pmatrix}, \quad \mathbf{u}_L^{(+)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_p+m} \end{pmatrix},$$

for solutions with positive energy $E = E_p$, and

$$\mathbf{u}_R^{(-)}(p) = \begin{pmatrix} \frac{-p}{E_p+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_L^{(-)}(p) = \begin{pmatrix} 0 \\ \frac{p}{E_p+m} \\ 0 \\ 1 \end{pmatrix},$$

for solutions with negative energy $E = -E_p$.

- What are the free-particle wave-functions?
- Calculate the four-current $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, where $\bar{\Psi} = \Psi^\dagger\beta$. Interpret your result.

*3. Chiral Symmetry

1+1+1+1+2+3 Points

(Note that for bachelor students this problem is not mandatory in the course with the lower number of credit points.)

Define the fifth γ -matrix as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and consider the Dirac Hamiltonian

$$H_D = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m,$$

with

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.$$

- Show that $\{\gamma^\mu \partial_\mu, \gamma^5\} = 0$. The first term in the anti-commutator is known as the Dirac operator. Since the Dirac Hamiltonian can be constructed using $\gamma^0\gamma^i = \alpha^i$ and $\gamma^0 = \beta$, the Hamiltonian anti-commutes with the operator $i\gamma^1\gamma^2\gamma^3$.
- Show that $(\gamma^5)^2 = \mathbb{1}_4$.
- Using that $\sigma^l\sigma^m = i\varepsilon_{lmk}\sigma^k + \delta_{lm}\mathbb{1}_2$ show that $\gamma^l\gamma^m = -i\varepsilon_{lmk}\Sigma^k - \delta_{lm}\mathbb{1}_4$ for $l, m = 1, 2, 3$ (this generalizes Problem 1(b)). Here $\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$.
- Consider now an operator \hat{C} with the property that $\hat{C}^2 = \mathbb{1}$ and $\{\hat{H}, \hat{C}\} = 0$. Show that if $|E_n\rangle$ is an eigenstate of the Hamiltonian H with eigenvalue E_n , then $|-E_n\rangle = \hat{C}|E_n\rangle$ is also an eigenstate of the Hamiltonian with eigenvalue $-E_n$.
- Consider now a two-level system with energy eigenvalues $\pm E_n$. Write down the matrix representations of \hat{C} and \hat{H} , and show that H is anti-diagonal in the basis where C is diagonal.
- Generalize your result in (e) to N non-degenerate levels. That is, show that it is possible to diagonalize C in such a way that H becomes anti-diagonal. What happens qualitatively when there are degenerate eigenstates?

Hint: Diagonalize C (you know its eigenvalues!). You can construct H using your result in (e). Write down a suitable basis. Think about how to rearrange the rows and columns of your matrices such that the diagonal elements in C are sorted with the positive eigenvalues coming before the negative eigenvalues.