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Advanced Quantum Mechanics - Problem Set 9

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 03.01.2022, 12:00**. The problem set will be discussed in the tutorials on Wednesday, 05.01.2022, and Friday, 07.01.2022.

*1. Addition of angular momenta

3+3+1 Points

Consider two angular momenta $\hat{\boldsymbol{L}}_1$ and $\hat{\boldsymbol{L}}_2$ with $l_1 = l_2 = 1$. In this problem we will calculate the eigenvalues and eigenfunctions of $\hat{\boldsymbol{L}}^2$. The eigenfunctions are linear combinations of the 9 functions

$$Y_{1m}(\theta_1, \varphi_1) Y_{1m'}(\theta_2, \varphi_2) = u_m v_{m'}, \qquad \text{with } m, m' = 1, 0, -1.$$

(a) Construct the 9×9 matrix representation of the operator $\hat{\boldsymbol{L}}^2$ in the $u_m v_{m'}$ basis.

(b) Calculate the eigenvalues of $\hat{\boldsymbol{L}}^2$ by diagonalizing the matrix.

(c) Calculate the corresponding eigenfunctions.

Hint: It is possible to make the matrix block-diagonal, as shown in the figure, by making suitable row- and column-operations.

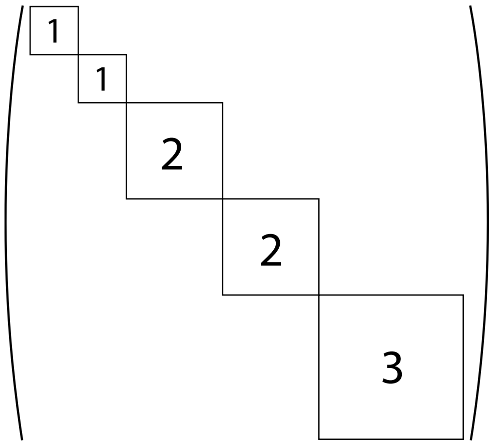

Figure 1: The matrix can be transformed into a block diagonal form.

2. Spin-orbit coupling

2+2+2 Points

Consider a particle with orbital angular momentum $\hat{\boldsymbol{L}}$ and spin angular momentum $\hat{\boldsymbol{S}}$. The total angular momentum is $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$.

(a) Calculate the expectation value of $\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}$ assuming that the particle is in a state $|l, s; j, m\rangle$.

(b) An electron is moving in an electrostatic potential $\phi(r)$ with $r = |\boldsymbol{r}|$. Show that the electric field experienced by the particle is given by

$$\boldsymbol{E} = -\boldsymbol{r}\frac{1}{r}\frac{d\phi}{dr}.$$

(c) In the rest frame of the particle, the particle experiences a magnetic field $\boldsymbol{B} = -\boldsymbol{v} \times \boldsymbol{E}/c^2$. Calculate the energy $\frac{e}{m}\hat{\boldsymbol{S}} \cdot \boldsymbol{B}$, where e and m are the electron charge and mass respectively.

Remark: The result found in (c) is off by a factor of two compared to the exact result, which can be obtained using the Dirac equation. The reason is that the simple argument given above assumes a straight-line motion of the particle whereas the potential given above leads to a curved particle trajectory.

3. Spin-orbit coupling in Hydrogen

4+2+2 Points

The spin-orbit Hamiltonian for Hydrogen is given by

$$H_{\text{SO}} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{m^2c^2r^3}\,\hat{\boldsymbol{S}} \cdot \hat{\boldsymbol{L}}.$$

We will treat this Hamiltonian as a perturbation in this problem.

(a) Using the relevant Hydrogen wave-function, calculate the leading order energy correction due to spin-orbit coupling, for $n = 2$, and $l = 1$. Take $s = 1/2$ as the spin of the electron.

(b) Use Kramers' relation

$$\frac{\alpha+1}{n^2}\langle r^\alpha\rangle - (2\alpha+1)a\langle r^{\alpha-1}\rangle + \frac{\alpha}{4}\left[(2l+1)^2 - \alpha^2\right]a^2\langle r^{\alpha-2}\rangle = 0,$$

where a is the Bohr radius, to derive a relation between $\langle r^{-2}\rangle$ and $\langle r^{-3}\rangle$.

(c) Calculate the leading order energy correction due to spin-orbit coupling for general n and l. You may use that

$$\langle r^{-2}\rangle = \frac{1}{(l+1/2)n^3a^2}.$$

Hint: The result of task 2 (a) might be helpful.