Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
Dr. S. Fischer

Advanced Quantum Mechanics - Problem Set 12

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 24.01.2022, 12:00**. The problem set will be discussed in the tutorials on Wednesday, 26.01.2022, and Friday, 28.01.2022

1. Number operator *4 Points*

Consider an operator $\hat{a}$ which satisfies $\{\hat{a}, \hat{a}^\dagger\} = \hat{a}\hat{a}^\dagger + \hat{a}^\dagger a = 1$ and $\{a, a\} = \{a^\dagger, a^\dagger\} = 0$. Show that the operator $\hat{N} = \hat{a}^\dagger \hat{a}$ has eigenvalues 0 and 1. What would you get if the anti-commutator is replaced by a commutator?

*2. Berry phase and the Aharonov-Bohm effect *2+2+1+3 Points*

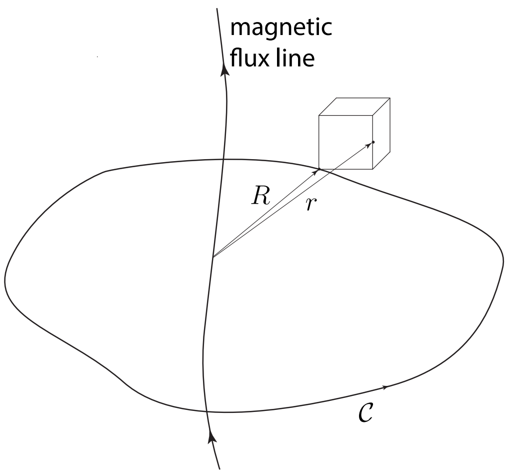

Figure 1: An electron in a box moves around a magnetic flux line. The path of the electron encloses a flux Φ_B.

Consider an electron in a small box moving along a closed loop $\mathcal{C}$, which encloses a magnetic flux Φ_B as shown in Fig. 1. Let $\boldsymbol{R}$ denote the position vector of a point on the box and $\boldsymbol{r}$ the position vector of the electron itself.

(a) Show that if the wave function of the electron in the absence of a magnetic field is $\psi_n(\boldsymbol{r} - \boldsymbol{R})$, then the wave function of the electron in the box at position $\boldsymbol{r}$ is

$$\langle \boldsymbol{r} | n(\boldsymbol{R}) \rangle = \exp\left[\frac{ie}{\hbar} \int_{\boldsymbol{R}}^{\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r'}) \cdot d\boldsymbol{r'}\right] \psi_n(\boldsymbol{r} - \boldsymbol{R}).$$

Here $\boldsymbol{A}$ denotes the vector potential. Note that this is only true if the magnetic field inside the box is zero. Why?

(b) Show that

$$\langle n(\boldsymbol{R})|\nabla_{\boldsymbol{R}}|n(\boldsymbol{R})\rangle = -\frac{ie}{\hbar}\boldsymbol{A}(\boldsymbol{R}).$$

(c) Calculate the geometric phase

$$\gamma_n(\mathcal{C}) = i\oint_{\mathcal{C}} \langle n(\boldsymbol{R})|\nabla_{\boldsymbol{R}}|n(\boldsymbol{R})\rangle \cdot d\boldsymbol{R},$$

and comment on your result.

(d) Suppose now an electron passes above or below a very long impenetrable cylinder, as shown in Fig. 2. Inside of the cylinder there is a magnetic field parallel to the cylinder axis, taken to be normal to the plane of the figure. Outside of the cylinder there is no magnetic field; the particle paths, however, enclose magnetic flux. Determine the phase associated with this magnetic flux for each path, and show how the resulting phase difference between the two paths affects the interference pattern.

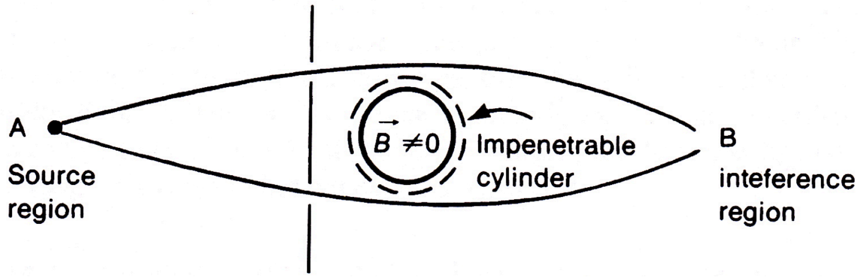

Figure 2: An electron moves either above or below an impenetrable cylinder enclosing a magnetic field parallel to its axis.

3. Anyons and the Aharonov-Bohm effect *2+2 Points*

Consider a two-dimensional electron gas in the presence of a magnetic field. The conductivity tensor is given by

$$\sigma = \begin{pmatrix} 0 & \sigma_{xy} \\ -\sigma_{xy} & 0 \end{pmatrix},$$

where $\sigma_{xy} = \nu e^2/h$, with $0 < \nu < 1$, is the Hall conductivity.

(a) Suppose now a flux Φ is switched on adiabatically. Using Faraday's law, and using that the current density is given by $\boldsymbol{J} = \sigma\boldsymbol{E}$, where $\boldsymbol{E}$ is the induced electric field, show that the charge satisfies $\dot{Q} = \sigma_{xy}\dot{\Phi}$. How does the charge change if the flux changes by $\Phi_0 = h/e$?

(b) Now consider the composite object (quasiparticle) of a flux Φ_0 and charge $q = \nu e$. Determine the mutual statistics of these quasiparticles. In which case do these composite objects behave as electrons? What do you get for $\nu = 1/3$ and $\nu = 1/5$? These states have been observed in experiments.

Hint: Exchange of the two quasiparticles corresponds to moving one quasiparticle by half a circle and performing a translation. This suggests that the wave function acquires a phase, which is half of the Berry phase acquired by a charge $q = \nu e$ moving along a path enclosing a magnetic flux Φ_0 (see problem 2). Use this line of reasoning to obtain the exchange statistics of the composite objects for different values of ν. Quasiparticles which acquire a phase different from 0 (bosons) or π (fermions) in the exchange are called anyons.

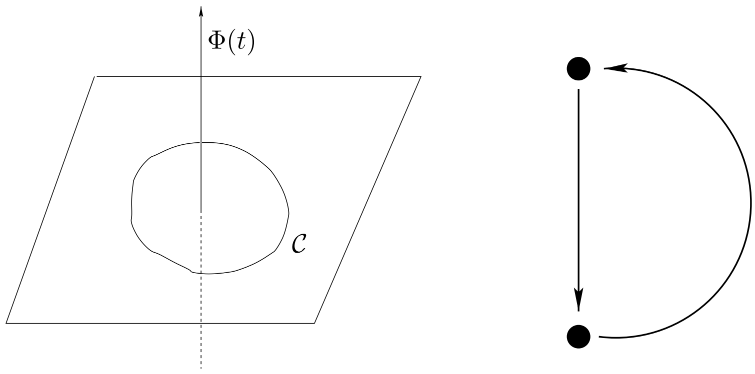

Figure 3: Left: The composite object is made up of a flux enclosed by a path $\mathcal{C}$ and a charge. Right: Illustration of how to exchange two quasiparticles.