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Advanced Quantum Mechanics - Problem Set 13

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 31.01.2022, 12:00**. The problem set will be discussed in the tutorials on Wednesday, 2.02.2022, and Friday, 2.02.2022

1. SSH model with a domain wall *3+3 Points*

In an earlier problem we considered the SSH model as an example of a tight-binding Hamiltonian. We found that the Hamiltonian can be written as

$$H(k) = \begin{pmatrix} 0 & -\gamma(k) \\ -\gamma^*(k) & 0 \end{pmatrix},$$

where $\gamma(k) = t + se^{-ik}$. Here t denotes the coupling within a unit cell and s the coupling between sites in different unit cells.

(a) Expand $\gamma(k)$ around a zone boundary $k = \pm\pi + q$ and thus show that the Hamiltonian becomes

$$H = m\sigma_x - i\partial_x s\sigma_y.$$

Give an expression for m in terms of s and t.

(b) Assume now that there is a domain wall at $x = 0$ such that $m(x) = m_0 \mathrm{sgn}(x)$. Find the zero-energy solution by demanding that the solutions are continuous at $x = 0$ and normalizable.
Hint: You may wish to apply H to the Schrödinger equation before solving.

*2. Unit cell in the presence of a magnetic field *2+5+3 Points*

Recall that the operator $\hat{T}_{\boldsymbol{a}} = e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\boldsymbol{p}}}$ is the generator of translations. For a Hamiltonian with lattice translation symmetry, these operators commute with the Hamiltonian. In a magnetic field this is no longer the case since the vector potential is not translationally invariant. In this problem we will consider a two-dimensional electron gas in the presence of a magnetic field in the z-direction $\boldsymbol{B} = (0, 0, B)$. The Hamiltonian can be written as

$$\hat{H} = \frac{(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))^2}{2m} + V(\boldsymbol{r}),$$

where $V(\boldsymbol{r})$ is the periodic lattice potential, i.e. $V(\boldsymbol{r} + \boldsymbol{a}) = V(\boldsymbol{r})$ for lattice vectors $\boldsymbol{a}$. For this problem we use the symmetric gauge $\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}(-By, Bx, 0)$.

(a) Show that the translation operator

$$\hat{\mathcal{T}}_{\boldsymbol{a}} = \exp\left\{\frac{i}{\hbar}\,\boldsymbol{a}\cdot[\hat{\boldsymbol{p}} + e\boldsymbol{A}(\boldsymbol{r})]\right\}$$

commutes with the Hamiltonian. This translation operator is called a magnetic translation operator.

(b) Show that

$$\hat{\mathcal{T}}_{\boldsymbol{a}}\hat{\mathcal{T}}_{\boldsymbol{b}} = \exp\left[\frac{i}{l_0^2}(\boldsymbol{a}\times\boldsymbol{b})\cdot\hat{\boldsymbol{e}}_z\right]\hat{\mathcal{T}}_{\boldsymbol{b}}\hat{\mathcal{T}}_{\boldsymbol{a}}.$$

Here $l_0 = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length and $\hat{\boldsymbol{e}}_z$ is a unit vector perpendicular to the plane.

(c) We now want to determine the enlarged unit cell such that the magnetic translation operators commute with each other. Let therefore $n\boldsymbol{a}$ and $m\boldsymbol{b}$ span an enlarged unit cell in the plane. In this case the magnetic translation operators have to commute with each other. Show that this is only possible if the flux $\Phi = \boldsymbol{B}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ satisfies

$$\frac{\Phi}{\Phi_0} = \frac{l}{mn},$$

with l an integer and $\Phi_0 = h/e$.