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Advanced Quantum Mechanics - Problem Set 6

Winter Term 2022/23

Due Date: Hand in solutions to problems marked with * to mailbox 39 with label “Advanced Quantum Mechanics” inside ITP room 105b before the lecture on **Friday, 25.11.2022, 09:15**. The problem set will be discussed in the tutorials on Monday 28.11.2022 and Wednesday 30.11.2022.

*1. Relativistic Landau Levels

3+3+2 Points

The low-energy Hamiltonian for electrons in graphene is given by

$$H = v_F \begin{pmatrix} -\boldsymbol{\sigma}^* \cdot \boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \boldsymbol{p} \end{pmatrix},$$

where v_F is the Fermi velocity, $\boldsymbol{p}$ the momentum and $\boldsymbol{\sigma}$ the vector of Pauli matrices. The eigenstates can be written as four-dimensional state vectors with contributions from the $\boldsymbol{K}$ and $\boldsymbol{K}'$ points. That is, we write

$$\boldsymbol{\chi} = \begin{pmatrix} \chi'_A \\ \chi'_B \\ \chi_A \\ \chi_B \end{pmatrix}.$$

(a) Show that the eigenvalue equations decouple into

$$E^2\chi_A = v_F^2(p_x - ip_y)(p_x + ip_y)\chi_A,$$
$$E^2\chi_B = v_F^2(p_x + ip_y)(p_x - ip_y)\chi_B,$$

and similar for the primed parts of the eigenstates.

(b) Suppose now a magnetic field is switched on. Using the Landau gauge $\boldsymbol{A} = (-By, 0)$, perform the minimal substitution $\boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$ in the eigenvalue equations in part (a) and deduce the form of the eigenfunctions.

(c) What does the energy spectrum look like?

2. Representations of γ matrices

2+1+2 Points

The γ matrices can be written as

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3$$
$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where σ_i denotes a Pauli matrix and $\mathbb{1}_n$ the $n \times n$ unit matrix. Consider the metric $\eta = \mathrm{diag}(1, -1, -1, -1)$.

(a) Show that the γ matrices satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4$, $\mu, \nu \in \{0, 1, 2, 3\}$.

(b) A different representation is the Weyl representation where

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

Show that these still satisfy the Clifford algebra.

(c) Using only the Clifford algebra and properties of the trace show that $\mathrm{tr}(\gamma_\mu) = 0$, $\mathrm{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}$, and $\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 0$.

3. Continuity equation for the Dirac equation *5 Points*

Prove the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0,$$

with

$$\boldsymbol{j} = \Psi^\dagger \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \Psi,$$

and $\rho = \Psi^\dagger\Psi$ for all solutions Ψ of the Dirac equation.