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Advanced Quantum Mechanics - Problem Set 14

Winter Term 2023/24

Due Date: Problems on this exercise sheet are **not** mandatory. Instead, points scored here are counted on top of points already reached for completing mandatory exercises. Hand in solutions to problems marked with * before **Wednesday, 31.01.2024, 15:15.** We can only correct solutions for students that are missing points for admittance to the exam.

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

*1. Unit cell in the presence of a magnetic field *2+5+3 Points*

Recall that the operator $\hat{T}_{\boldsymbol{a}} = e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\boldsymbol{p}}}$ is the generator of translations. For a Hamiltonian with lattice translation symmetry, these operators commute with the Hamiltonian. In a magnetic field this is no longer the case since the vector potential is not translationally invariant. In this problem we will consider a two-dimensional electron gas in the presence of a magnetic field in the z-direction $\boldsymbol{B} = (0, 0, B)$. The Hamiltonian can be written as

$$\hat{H} = \frac{(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))^2}{2m} + V(\boldsymbol{r}),$$

where $V(\boldsymbol{r})$ is the periodic lattice potential, i.e. $V(\boldsymbol{r}+\boldsymbol{a}) = V(\boldsymbol{r})$ for lattice vectors $\boldsymbol{a}$. For this problem we use the symmetric gauge $\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}(-By, Bx, 0)$.

(a) Show that the translation operator

$$\hat{\mathcal{T}}_{\boldsymbol{a}} = \exp\left\{\frac{i}{\hbar}\boldsymbol{a}\cdot[\hat{\boldsymbol{p}} + e\boldsymbol{A}(\boldsymbol{r})]\right\}$$

commutes with the Hamiltonian. This translation operator is called a magnetic translation operator.

(b) Show that

$$\hat{\mathcal{T}}_{\boldsymbol{a}}\hat{\mathcal{T}}_{\boldsymbol{b}} = \exp\left[\frac{i}{l_0^2}(\boldsymbol{a}\times\boldsymbol{b})\cdot\hat{\boldsymbol{e}}_z\right]\hat{\mathcal{T}}_{\boldsymbol{b}}\hat{\mathcal{T}}_{\boldsymbol{a}}.$$

Here $l_0 = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length and $\hat{\boldsymbol{e}}_z$ is a unit vector perpendicular to the plane.

(c) We now want to determine the enlarged unit cell such that the magnetic translation operators commute with each other. Let therefore $n\boldsymbol{a}$ and $m\boldsymbol{b}$ span an enlarged unit cell in the plane. In this case the magnetic translation operators have to commute with each other. Show that this is only possible if the flux $\Phi = \boldsymbol{B}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ satisfies

$$\frac{\Phi}{\Phi_0} = \frac{l}{mn},$$

with l an integer and $\Phi_0 = h/e$.

*2. Zitterbewegung

2+2+2 Points

In this problem we will consider the Dirac Hamiltonian

$$\hat{H}_D = c\boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta mc^2,$$

where m is the mass of the particle, c is the speed of light, and $\boldsymbol{\alpha}$ and β are matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

with $\boldsymbol{\sigma}$ denoting the vector of Pauli matrices and I_2 denoting the 2×2 unit matrix. The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the velocity operator is given by $\hat{\boldsymbol{v}} = c\boldsymbol{\alpha}$.

Hint: You may use the Heisenberg equation of motion which states that an operator $\hat{A}$ which does not explicitly depend on time satisfies $-i\hbar\dot{\hat{A}} = [\hat{H}, \hat{A}]$.

(b) Consider now a Dirac particle at rest in a volume V. A general eigenspinor can then be written as

$$\psi = \frac{1}{\sqrt{2V}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2t/\hbar} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imc^2t/\hbar} \right].$$

Give a physical interpretation of the two terms in the spinor.

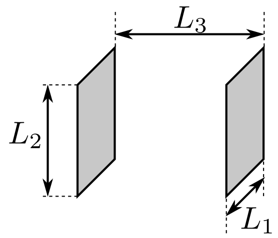
(c) Derive an expression for $\langle \hat{v}_z \rangle = \langle \psi | \hat{v}_z | \psi \rangle$ using the spinor defined in the previous part of the problem. Comment on your result.

*3. Casimir Effect

4+4+2+1 Points

As shown in Problem Set 10 Exercise 2, the Hamiltonian of the quantized radiation field confined to a box with volume $V = L_1 L_2 L_3$ and with periodic boundary conditions, is given by

$$H = \sum_{\boldsymbol{k}} \sum_{\lambda=\pm} \hbar\omega_{\boldsymbol{k}} \left(a^\dagger_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda} + \frac{1}{2} \right) , \quad \omega_{\boldsymbol{k}} = c|\boldsymbol{k}| , \quad k_i = \frac{\pi}{L_i} n_i , \quad n_i \in \mathbb{N} .$$



In particular we found that the ground state, in which no modes are excited, has a divergent energy. Whilst this divergent vacuum zero-point energy is not observable, the dependence on the boundaries does lead to observable phenomena.
To investigate this, we consider in the following two conducting plates with surface areas $A = L_1 L_2$ separated by a distance L_3. In the plane of the plates we will still be using periodic boundary conditions and consider the limit $L_1, L_2 \to \infty$. Since the electric field $\boldsymbol{E}$ on the plates vanishes, only modes with $|\boldsymbol{E}| \propto \sin(k_3 x_3)$ are possible. Here $k_3 = n_3\pi/L_3$ with $n_3 = 1, 2, \ldots$. To get a finite vacuum energy we will moreover introduce an exponential cutoff $e^{-\epsilon\omega_{\boldsymbol{k}}}$ with $\epsilon > 0$, and take the limit of $\epsilon \to 0$ at the end of the calculation. The energy density per unit plate area between the plates is given by

$$\begin{aligned} \sigma_E(L_3) &= \lim_{L_1,L_2\to\infty} \frac{1}{L_1 L_2} \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}}\, e^{-\epsilon\omega_{\boldsymbol{k}}} \\ &= \hbar c \sum_{n_3=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \sqrt{k_1^2 + k_2^2 + (\tfrac{\pi n_3}{L_3})^2}\, e^{-\epsilon c\sqrt{k_1^2+k_2^2+(\frac{\pi n_3}{L_3})^2}} \end{aligned}$$

(a) Using polar coordinates and a suitable subsitution show that $\sigma_E(L_3)$ can be written as

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \sum_{n=1}^{\infty} \int_{n\pi c/L_3}^{\infty} d\omega e^{-\epsilon\omega} .$$

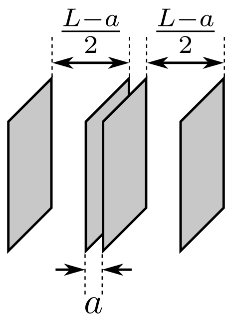
(b) Calculate the integral over ω and perform the sum to show that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \left(\frac{1}{\epsilon} \frac{1}{e^{\epsilon\pi c/L_3} - 1} \right) .$$

Show further that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \left(\frac{6}{\epsilon^4} \frac{L_3}{\pi c} - \frac{1}{\epsilon^3} - \frac{1}{360} \left(\frac{\pi c}{L_3} \right)^3 + \mathcal{O}(\epsilon^2) \right).$$

(c) The energy density calculated in the previous part diverges as the distance between the plates increases ($L_3 \to \infty$). This will be our reference point. We therefore consider two plates separated by a fixed distance a, together with two external plates which are places a further distance $(L-a)/2$ away. The relevant energy density is then given by



$$\sigma_E(a, L) = \sigma_E(a) + 2\sigma_E \left(\frac{L-a}{2} \right) .$$

Find an expression for $\sigma_E(a, L)$ using your result in (b).

(d) Since the energy density varies with the distance between plates, the plates experience a pressure which is given by

$$p_{\text{vac}} = -\lim_{L\to\infty} \frac{\partial}{\partial a}\sigma_E(a, L).$$

How large is this pressure for $A = 1\,\text{cm}^2$ and $a = 1\,\mu\text{m}$?