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Advanced Statistical Physics - Problem Set 7

Summer Term 2018

Due Date: Tuesday, May 29, 09:15 a.m., mailbox inside ITP

Internet: Advanced Statistical Physics exercises

The first problem is a mathematical problems, which will help you to get familiar with Fourier transformations. The second problem illustrates the use of the mean field approximation in the description of phase transitions.

8. Fourier transform *2 + 3 Points*

a) The Fourier transform of a real field $\psi(x)$ is defined as

$$\psi(x) = \frac{1}{\sqrt{L^d}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}}.$$

Show that the inverse transformation is given by

$$\psi_{\mathbf{k}} = \frac{1}{\sqrt{L^d}} \int d^d x \; e^{-i\mathbf{k}\cdot\mathbf{x}} \psi(x) = \alpha_{\mathbf{k}} + i\beta_{\mathbf{k}},$$

and further deduce that $\alpha_{\mathbf{k}} = \alpha_{-\mathbf{k}}$ and $\beta_{\mathbf{k}} = -\beta_{-\mathbf{k}}$.

b) Explicitly derive the Fourier transform of the Landau-Ginzburg Hamiltonian

$$\mathcal{H} = \int d^d x \left[\frac{a\tau}{2}\psi^2(x) + \frac{u}{4}\psi^4(x) + \frac{c}{2}(\nabla\psi)^2 - h(x)\psi(x) \right].$$

9. Domain wall picture of Ising phase transition *2 + 2 + 2 + 1 + 2 Points*

The thermodynamic limit is a mathematical necessity to observe the singular behavior of susceptibilities or correlations lengths. For systems of finite size, the free energy can be expressed as a finite order polynomial of the order parameter and its derivatives. Thus, no phase transition in the mathematical sense can be observed. The aim of this exercise is to show that if one includes dynamics, the thermodynamic limit is not necessary to observe phase transitions.
Consider a two-dimensional lattice with periodic boundary conditions of N spins with interaction constant J which is in the ordered phase for $T < T_c$. Here, either the state with magnetization $m = -1$ or $m = +1$ is spontaneously realized. To switch between them, the formation of a domain wall is needed as shown in the figure on the next page.
The aim of the task is to estimate the critical temperature T_c and the number of spins N that is actually needed in order to get a physically stable thermodynamic phase.

a) Verify that the change in free energy ΔF_{wall} of the domain wall of length L can be estimated as

$$\Delta F_{\text{wall}} = L\left(2J - k_B T \ln 3\right) \ .$$

Hint: To calculate the internal energy, estimate the energy that is needed to flip one spin. The entropy is given by $S = k_B \ln W$, with W being the number of possible realizations of domain walls with length L. At each lattice site, there are three possible directions for the domain wall.

b) Argue that a phase transition occurs at $\Delta F_{\text{wall}} = 0$. Determine the critical temperature and compare to the Onsager solution for the 2D Ising model

$$k_B T_c^{\text{Onsager}} = \frac{2J}{\ln\left(1+\sqrt{2}\right)} \ .$$

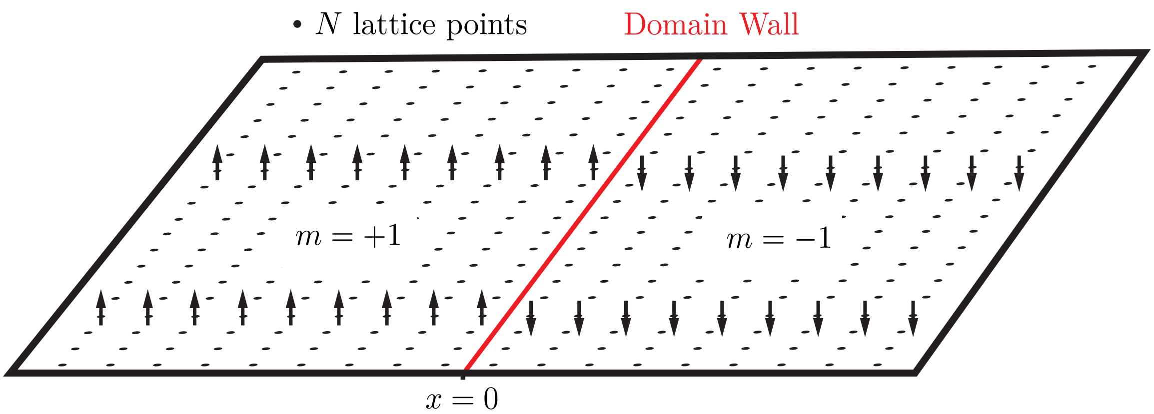
Comment on the validity of your estimation.

c) Now argue that for the formation of a domain wall separating the system into two distinct regions we must have $L \geq \sqrt{N}$, and determine ΔF_{wall} for $T = T_c/2$ and $L = \sqrt{N}$.

d) Finally, assume that the dynamics follows the Arrhenius law

$$\tau = \tau_0 \cdot e^{\beta \Delta F_{\text{wall}}} \ ,$$

with $\tau_0 = 10^{-12}\,\text{s}$ being the characteristic microscopic spin-flip time. Again working with $T = T_c/2$ estimate the number of spins, which are necessary to spontaneously flip the magnetization within the lifetime of the universe $\tau = 4,3 \cdot 10^{17}\,\text{s}$.



Domain wall separating two regions of different magnetization in the ordered phase.