Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
N. John, M. Thamm

Advanced Statistical Physics - Problem Set 10

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Due Date: Tuesday, June 19, 09:15 a.m., mailbox inside ITP

Internet: Advanced Statistical Physics exercises

14. Cooper pair size

2+4+1 Points

a) The Cooper pair wave function can be expanded in a plane-wave basis as

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} g(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{r}} \ ,$$

with $g(\boldsymbol{k})$ being the amplitude of finding an electron in state with momentum $\boldsymbol{k}$ and another one in a state with momentum $-\boldsymbol{k}$, and $\boldsymbol{r}$ being the relative coordinate (distance between the electrons) of the Cooper pair. Note that $g(\boldsymbol{k}) = 0$ for $k < k_F$. Start with the definition of the mean square radius

$$R^2 = \frac{\int d\boldsymbol{r}\, r^2 |\psi(\boldsymbol{r})|^2}{\int d\boldsymbol{r}\, |\psi(\boldsymbol{r})|^2} \ ,$$

and show that it can be reexpressed as

$$R^2 = \frac{\sum_{\boldsymbol{k}} |\nabla_{\boldsymbol{k}}\, g(\boldsymbol{k})|^2}{\sum_{\boldsymbol{k}} |g(\boldsymbol{k})|^2} \ .$$

b) By turning the sums into integrals and by using the definition of the density of state, show that the expression for the mean square radius becomes

$$R^2 \simeq \frac{\left(\frac{\partial \xi}{\partial k}\right)^2_{\xi=0} \int\limits_0^\infty d\xi \left(\frac{\partial g(\xi)}{\partial \xi}\right)^2}{\int\limits_0^\infty d\xi\, g(\xi)^2} \ .$$

Further, use that $g(\xi) \propto 1/(\Delta + 2\xi)$ to show that the mean square radius of a Cooper pair is given by

$$R = \frac{2}{\sqrt{3}} \frac{\hbar v_F}{\Delta} \ ,$$

with v_F being the Fermi velocity, and Δ being the binding energy of the Cooper pair relative to the Fermi surface.

c) Insert realistic values for v_F and Δ and estimate the size of the Cooper pair.

15. Specific heat exponent and scaling relation *4+4 Points*

a) Calculate the specific heat critical exponent using

$$C_{\text{sing}}(t,h) = -T\frac{\partial^2}{\partial T^2} f_{\text{sing}}(t,h) \ ,$$

and the scaling hypothesis for $f_{\text{sing}}(t,h)$. Start from the generalized homogeneity equation

$$\lambda f_{\text{sing}}(t,h) = f_{\text{sing}}(\lambda^{a_t} t, \lambda^{a_h} h) \ .$$

Hint: Use an appropriate expression for λ to obtain the form of the singular part of the free energy as given in the lectures

$$f_{\text{sing}}(t,h) = |t|^c g_{f,\pm}(t/h^{\Delta}) \ .$$

Use the scaling of C_{sing} to relate c and α.

b) Exactly at the critical point, the correlation length is infinite, and therefore all correlations decay as a power-law of the separation. From scattering experiments one obtains that

$$\langle m(\boldsymbol{x}) m(0)\rangle_c \sim 1/|\boldsymbol{x}|^{d-2+\eta} \ .$$

Away from criticality, the correlation functions decay exponentially with the length scale determined by the correlation length $\xi(t,h)$. This exponential decay can be approximated with an abrupt decay of the correlation function to zero at $|\boldsymbol{x}| \sim \xi(t,h)$.

Use the definition of the susceptibility

$$\chi \sim \int d^d x \langle m(\boldsymbol{x}) m(0)\rangle_c$$

to derive Fisher's identity, which establishes a connection between the correlation length exponent ν, the correlation function exponent η, and the susceptibility exponent γ.