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Advanced Statistical Physics - Problem Set 5

Summer Term 2019

Due Date: Wednesday, May 15, 12:00 p.m., Hand in tasks marked with * to mailbox with label 'Advanced Statistical Physics Exercises' inside ITP room 105b

Internet: Advanced Statistical Physics exercises

6. BCS as a mean-field theory

4+4+4+4 Points

In this problem we will consider electrons interacting through the BCS interaction, which is non-zero in a shell around the Fermi surface. We consider the following Hamiltonian

$$H = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}\boldsymbol{k}'} V(\boldsymbol{k}, \boldsymbol{k}') c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \ ,$$

where

$$V(\boldsymbol{k}, \boldsymbol{k}') = \begin{cases} -V/\Omega, & |\xi_{\boldsymbol{k}}|, |\xi_{\boldsymbol{k}'}| \leq \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases} .$$

Here, $\xi_{\boldsymbol{k}} = \hbar^2 \boldsymbol{k}^2/2m - E_F$, and ω_D denotes the Debye frequency. Motivated by the condensation into momentum space pairs $(\boldsymbol{k}\uparrow, \boldsymbol{k}\downarrow)$, we introduce a mean field expectation value $\langle c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} \rangle$, which is called condensation amplitude when computed with respect to the BCS ground state $|\phi\rangle$.

a) * Proceeding analogously to Hartree-Fock mean-field theory as shown in the lectures, convince yourself that the mean-field decoupling of the Hamiltonian yields

$$H = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k}} \Delta_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} - \sum_{\boldsymbol{k}} \Delta^*_{\boldsymbol{k}} c_{\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\uparrow} - \sum_{\boldsymbol{k}\boldsymbol{k}'} V(\boldsymbol{k}, \boldsymbol{k}') \langle c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \rangle \langle c_{\boldsymbol{k}'\downarrow} c_{-\boldsymbol{k}'\uparrow} \rangle \ ,$$

with the self-consistency condition given by

$$\Delta_{\boldsymbol{k}} = -\sum_{\boldsymbol{k}'} V(\boldsymbol{k}, \boldsymbol{k}') \langle c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \rangle \ .$$

b) * Show that the Hamiltonian can be diagonalized by the following unitary transformation

$$\begin{pmatrix} \gamma_{\boldsymbol{k}\uparrow} \\ \gamma^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\boldsymbol{k}} & \sin\theta_{\boldsymbol{k}} \\ \sin\theta_{\boldsymbol{k}} & -\cos\theta_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} \ ,$$

such that

$$H = \sum_{\boldsymbol{k}\sigma} E(\boldsymbol{k}) \gamma^\dagger_{\boldsymbol{k}\sigma} \gamma_{\boldsymbol{k}\sigma} + \text{constant.}$$

Derive an expression for $E(\boldsymbol{k})$.

c) Show that at zero temperature the self-consistency condition is equivalent to the BCS gap equation. Next, use the self-consistency condition to find the transition temperature T_c for the case $T > 0$.

Hint: Use the unitary transformation obtained in part (b) to rewrite the product $c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow}$ in terms of $\gamma^{\dagger}_{\boldsymbol{k}\sigma}$ and $\gamma_{\boldsymbol{k}\sigma}$. Next, use that $\langle \gamma^{\dagger}_{\boldsymbol{k}\sigma}\gamma_{\boldsymbol{k}'\sigma'}\rangle = f_{\boldsymbol{k}} \cdot \delta_{\boldsymbol{k}\boldsymbol{k}'}\delta_{\sigma\sigma'}$, with $f_{\boldsymbol{k}}$ being the Fermi function as defined in exercise 5. In the case $T > 0$, you will end up with an integral which diverges logarithmically in the limit $T_c \to 0$. Use integration by parts to extract the divergent part. In order to solve the remaining (convergent) integral, you may use the limit $T_c \to 0$. The remaining integral will be given by

$$\int_0^\infty dx \frac{\ln x}{\cosh^2(x)} = -\gamma + \ln\left(\frac{\pi}{4}\right) ,$$

with $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right] \approx 0.5772$ being the Euler-Mascheroni constant.

d) *Bonus*: Compute the integral given above.

Hint: You may use that

$$\ln x = \lim_{n\to 0} \frac{x^n - 1}{n} ,$$

and you may change the order of integration and applying the limit $n \to 0$. In case your favourite computer algebra program fails to solve the integrals, consider looking them up in a table of integrals like the book by Gradshteyn and Ryshik.