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Advanced Statistical Physics - Problem Set 13

Summer Term 2019

Due Date: Wednesday, July 10, 12:00 p.m., Hand in tasks marked with * to mailbox with label 'Advanced Statistical Physics Exercises' inside ITP room 105b

Internet: Advanced Statistical Physics exercises

20. Kosterlitz–Thouless transition*

4+2+3 Points

The Kosterlitz–Thouless transition is a second order phase transition in the two-dimensional xy-model due to topological defects. We describe $\theta(\boldsymbol{x})$ as the spatial orientation of a spin-density $\boldsymbol{m}(\boldsymbol{x}) = \overline{m}\,(\cos\theta(\boldsymbol{x})\boldsymbol{e}_x + \sin\theta(\boldsymbol{x})\boldsymbol{e}_y)$. We require $\boldsymbol{m}(\boldsymbol{x})$ to be continuous everywhere except at a single point and consider a vortex with

$$\theta(r,\varphi) = n\varphi + \theta_0$$

where r and φ are polar coordinates and n is an integer. The singularity at the origin is removed by requiring $\langle \boldsymbol{m}(\boldsymbol{x})\rangle = 0$ inside the vortex core of size a, centered around the origin.

a) Using the definition of $\theta(r,\varphi)$, explicitly calculate the integral

$$\frac{1}{2\pi}\int_{\mathcal{C}} \mathrm{d}\boldsymbol{l}\cdot\nabla\theta$$

where $\mathcal{C}$ is a circle with radius R around the origin. Give an interpretation for the meaning of the variable n. Comment on the behavior of the angle θ at the positive x-axis, i.e. on $\theta(x,0)$? How is this compatible with continuity of the order parameter $\boldsymbol{m}(\boldsymbol{x})$?

b) The energy of a vortex in a circular volume Ω with radius R is divided into two contributions. i) the energy E_c of the vortex core. The core is a circular region with radius a around the center of the vortex. ii) the elastic energy $E_\mathrm{el} = (K/2)\int \mathrm{d}^2x(\nabla\theta)^2$. Show that the elastic energy is given by

$$E_\mathrm{el} = \pi K n^2 \ln\left(\frac{R}{a}\right) .$$

c) Obtain an expression for the free energy $F = E_\mathrm{el} - TS$ of a vortex. Find the transition temperature T_c above which vortices can proliferate and destroy the ordered phase.

Hint: To obtain an expression for the entropy, you may want to estimate the number of different vortex positions in the area Ω by taking into account the finite area πa^2 of the vortex core. To find the transition temperature, consider at which temperature the contribution of the vortex to the free energy changes sign.

21. Coupled scalars

1+3+2+2+2 Points

Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d x \left[\frac{t}{2} m^2 + \frac{K}{2}(\nabla m)^2 - hm + \frac{L}{2}(\nabla^2 \phi)^2 + v(\nabla m)(\nabla \phi) \right] ,$$

coupling two one-component fields m and ϕ.

a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\boldsymbol{q})$ and $\phi(\boldsymbol{q})$.

b) Construct a renormalization group transformation by rescaling distances such that $\boldsymbol{q}' = b\boldsymbol{q}$, and the fields such that $m'(\boldsymbol{q}') = \tilde{m}(\boldsymbol{q})/z$ and $\phi'(\boldsymbol{q}') = \tilde{\phi}(\boldsymbol{q})/y$. You do not need to evaluate the integrals that just contribute a constant additive term.

c) There is a fixed point such that $K' = K$ and $L' = L$. Find y_t, y_h and y_v at this fixed point.

d) The singular part of the free energy has a scaling form

$$f(t, h, v) = t^{2-\alpha} g(h/t^{\Delta}, v/t^{\omega})$$

for t, h, v close to zero. Find α, Δ and ω.

e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point, and how do they scale?