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Advanced Statistical Physics - Problem Set 1

Summer Term 2020

Due Date: Tuesday, April 14, 10:00 a.m., solutions must be mailed to stp2.leipziguni@gmail.com

The aim of the problem set is to get familiar with the mathematical description of the phase transitions and to illustrate the singular behavior of the response functions in the vicinity of a critical point.

1. * The Ising model of magnetism

2+4+2+3+2 Points

The local environment of an electron in a crystal sometimes forces its spin to stay parallel or anti-parallel to a given lattice direction. As a model of magnetism in such materials we denote the direction of the spin by a single variable $\sigma_i = \pm 1$ (an Ising spin). The energy of a configuration $\{\sigma_i\}$ of spins is then given by

$$\mathcal{H} = \frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i,$$

where h is an external magnetic field, and J_{ij} is the interaction energy between spins at sites i and j.

(a) Assume that the interaction energy between all spins is the same, i.e. $J_{ij} = -J/N$ for all $i, j = 1, 2, ..., N$. Show that the energy of a spin configuration depends only on the magnetization and can be written as $E(m, h) = -N[Jm^2/2 + hm]$. The magnetization of a spin configuration is defined as $m = \sum_{i=1}^{N} \sigma_i/N$.

(b) Show that the partition function $Z(h,T) = \sum_{\{\sigma_i\}} \exp(-\beta\mathcal{H})$ can be rewritten as $Z = \sum_m \exp\left[-\beta F(m,h)\right]$. Expand $F(m,h)$ to 4th order in m, and show that it can be written as

$$\frac{F(m,h)}{N} = -k_BT\ln 2 - hm + \frac{1}{2}(k_BT - J)m^2 + \frac{k_BT}{12}m^4.$$

For the remainder of the problem work only with the expansion of $F(m,h)$. Hint: since the energy of a spin configuration $\{\sigma_i\}$ only depends on the magnetization of this configuration, you need to find the number of spin configurations with a given magnetization in order to calculate the partition function - this is a combinatorics problem which you may have encountered before. Use Stirling's approximation for large N ($\ln N! \approx N\ln N - N$).

(c) Show that in the limit $N \to \infty$ the actual free energy $F(h,T) = -k_BT\ln Z(h,T)$ is given by $F(h,T) = \min[F(m,h)]_m$. Hint: Assume that $\min[F(m,h)]_m = F(m^*,h)$. Using the relation $Z = \sum_m \exp\left[-\beta F(m,h)\right]$, show that $\exp\left[-\beta F(m^*,h)\right] \le Z \le N\exp\left[-\beta F(m^*,h)\right]$ and use this result to prove the statement.

(d) Use the definition of the actual free energy to show that the magnetization is given by

$$\overline{m} = -\frac{1}{N}\frac{\partial F(h,T)}{\partial h}.$$

Assume that $h = 0$, and use the results obtained in (b) and (c) to find the critical temperature T_c below which a spontaneous magnetization appears. Calculate the temperature dependence of the magnetization $\overline{m}(T)$.

(e) Calculate the singular (non-analytic) behavior of the response functions

$$C = \left.\frac{\partial E}{\partial T}\right|_{h=0}, \quad \text{and} \quad \chi = \left.\frac{\partial \overline{m}}{\partial h}\right|_{h=0}.$$