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Advanced Statistical Physics - Problem set 3

Summer Terms 2022

Hand in: Hand in tasks marked with * to mailbox no. (43) inside ITP room 105b by Friday 29.04. at 9:15 am.

4. Surface of N-dimensional sphere*

3+2+3 Points

(a) Calculate the surface area Ω of an $N-$ dimensional sphere ?

(b) Show that the surface of an $N-$sphere is for large N to leading order given by

$$\Omega_0 = \exp\left(\frac{N}{2}[1 + \ln 2\pi]\right).$$

(c) Evaluate the surface of a cap on the N-sphere defined by all vectors $\boldsymbol{J}$ that make an angle smaller or equal to θ with a given direction defined by the vector $\boldsymbol{T}$. Show that this surface is dominated by the rim of the cap ?

5. The annealed approximation*

4+4 Points

In the lectures, we have introduced the auxiliary variables

$$\lambda_\mu = \frac{1}{\sqrt{N}}\boldsymbol{J}\boldsymbol{\xi}^\mu \ , \ u_\mu = \frac{1}{\sqrt{N}}\boldsymbol{T}\boldsymbol{\xi}^\mu$$

(a) Show that the joint probability density $P(\lambda, u)$ is indeed a Gaussian probability density . Start from

$$P(\lambda, u) = \left\langle\left\langle \delta\left(\lambda - \frac{1}{\sqrt{N}}\boldsymbol{J}\boldsymbol{\xi}\right)\delta\left(u - \frac{1}{\sqrt{N}}\boldsymbol{T}\boldsymbol{\xi}\right)\right\rangle\right\rangle_{\boldsymbol{\xi}}$$

where the average is with respect to a randomly chosen example $\boldsymbol{\xi}$, which have the probability distribution function

$$P(\boldsymbol{\xi}) = \prod_j\left[\frac{1}{2}\delta(\boldsymbol{\xi}_j + 1) + \frac{1}{2}\delta(\boldsymbol{\xi}_j - 1)\right]$$

Hint : you may use
The integral representation of the delta function

$$\delta(x) = \int \frac{d\hat{x}}{2\pi}e^{i\hat{x}x}$$

The Hubbard-Stratonovich transformation

$$\int Dt \ e^{bt} = e^{b^2/2}$$

$$\text{where } Dt := \frac{dt}{\sqrt{2\pi}}\exp(-t^2/2)$$

(b) Show that the distribution $P(\lambda, u)$ has the moments

$$\begin{aligned} \langle\langle\lambda\rangle\rangle &= \langle\langle u\rangle\rangle = 0 \\ \langle\langle\lambda^2\rangle\rangle &= \langle\langle u^2\rangle\rangle = 1 \\ \langle\langle\lambda u\rangle\rangle &= \frac{\boldsymbol{J}\boldsymbol{T}}{N} = R. \end{aligned}$$